Let $A \in M_r(\mathbb{Z})$ be a symmetric, positive definite $r \times r$ matrix.
Last class we proved when $r \geq 1$

Theorem Let N be such that $NA^{-1} \in M_r(\mathbb{Z})$. Let diag A be even. Then for all $\tau \in \Gamma_0(2N)$ we have

$$\Theta(\tau z, A) = \left(\frac{\det A}{d}\right)\left(\varepsilon_d^{-1}\left(\frac{2c}{d}\right)(cz+d)^{1/2}\right)^r \Theta(z, A).$$

Here $\varepsilon_d = \begin{cases} 1 & d \equiv 1\ (4) \\ i & d \equiv 3\ (4) \end{cases}$

$\left(\frac{\cdot}{\cdot}\right)$ is the Jacobi symbol: $\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a square mod } p,\ p \nmid a \\ -1 & \text{if } a \text{ is } \underline{\text{not}} \text{ a square mod } p,\ p \nmid a \\ 0 & \text{if } p \mid a \end{cases}$

extend multiplicatively.

If r is even, we can simplify further.

Let $(-1)^{r/2} \det A = D$ the "Discriminant" of A.

Theorem If diag $A \equiv$ diag $NA^{-1} \equiv 0\ (2)$ and r even then

$$\Theta(\tau z, A) = \left(\frac{D}{d}\right)(cz+d)^{r/2}\Theta(z, A)$$

for all $\tau \in \Gamma_0(N)$.

We apply these results to Quadratic forms:

Let $Q(x) = \frac{1}{2}A[x] = \frac{1}{2}{}^t x A x$, $x \in \mathbb{Z}^r$ be the quadratic form associated to A.

If diag $A \equiv 0\ (2)$, then Q has integral coeffs.

Then $\Theta(z, A) = \sum_{x \in \mathbb{Z}^r} e\left(\frac{1}{2}A[x]z\right) = \sum_{n \geq 0} r(n, Q)\, e(nz)$

where $r(n,Q) = \#\{x \in \mathbb{Z}^r : Q(x) = n\}$.

~~Supposed~~ We proved in Lecture 12 that the above spaces of modular forms are finite dimensional.

Idea: Expand $\Theta(z,A)$ into Eisenstein series & cusp forms:

$$\Theta(z,Q) = E(z,Q) + F(z,Q)$$

where $F(z,Q)$ is a cusp form, and $E(z,Q)$ is a linear combination of eisenstein series.

We have

$r(n,Q) = \rho(n,Q) + \tau(n,Q)$ where $\rho(n,Q)$ are the F.C. of $E(z,Q)$ and $\tau(n,Q)$ are the F.C of $F(z,Q)$.

If eg. $r \geq 4$ is even, $E(z,Q)$ is a linear combination of

$\mathfrak{a} \in \mathrm{Cusp}(\Gamma_0(N))$

$$E_{\mathfrak{a}}(z) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \frac{\chi(\gamma)}{j(z, \sigma_{\mathfrak{a}}^{-1}\gamma)^{r/2}}$$

$\sigma_{\mathfrak{a}} \infty = \mathfrak{a}$

$j(z,\sigma) = cz+d$

$\Gamma_{\mathfrak{a}} = \mathrm{stab}_{\Gamma} \mathfrak{a}$

Recall in exercise 5.1 you ~~proved~~ computed the Fourier expansion of $E_{\mathfrak{a}}$ fairly explicitly. So by the prop at the end of lecture 6 we have $\tau(n,Q) \ll n^{r/4}$, so

$$r(n,Q) = \rho(n,Q) + O(n^{r/2}).$$

However, the coeffs of the linear combination $E(z,Q) = \sum_{\mathfrak{a}} \varphi_{\mathfrak{a}} E_{\mathfrak{a}}(z)$ aren't so easy to compute.

We can still make this strategy work nicely if $N = 1$.

Suppose A is such that $\det A = 1$, $A, A^{-1} \in M_r(\mathbb{Z})$, so $N=1$ and $\operatorname{diag} A \equiv \operatorname{diag} A^{-1} \equiv 0 \,(2)$

Do such A exist?

Recall last Thursday we showed under these conditions that
(First prop of Lecture 17)

$$\Theta(-1/z, A) = (-iz)^{r/2}\, \Theta(z, A).$$

So if $\Theta \in M_{r/2}(SL_2\mathbb{Z})$, we must have $(-i)^{r/2} = 1$, so ~~If we then~~ $r \equiv 0\,(8)$.

Minkowski gave an example:

$$A = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 4 & 3 & & & & \\ & & 3 & 4 & 5 & & & \\ & & & 5 & 20 & 3 & & \\ & & & & 5 & 12 & 1 & \\ & & & & & 3 & 4 & 1 \\ & & & & & & 1 & 2 \end{pmatrix}$$

has $\det A = 1$, $Q(x) = \frac{1}{2}A[x]$ has $\mathbb{Z}$-coefficients.
So by the Theorem $\Theta(z, A) \in M_4(SL_2\mathbb{Z})$ but $\dim M_4(SL_2\mathbb{Z}) = 1$.
$\langle E_4 \rangle = M_4(SL_2\mathbb{Z})$.

$$E_4(z) = 1 + 240 \sum_{n \geq 1} \sigma_3(n)\, e(nz), \text{ where } \sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}$$

$$\Rightarrow \quad r(n, Q) = 240\, \sigma_3(n).$$

More generally if A is any matrix satisfying the above conditions, $\Theta(z, A) \in M_{r/2}(SL_2\mathbb{Z})$

And $M_{r/2}(SL_2\mathbb{Z}) / S_{r/2}(SL_2\mathbb{Z})$ is 1-diml, spanned by

$$E_k(z) = 1 + \frac{(2\pi)^k}{\zeta(k)\Gamma(k)} \sum_{n \geq 1} \sigma_{k-1}(n)\, e(nz) \qquad \text{(Lecture 5)}$$

Since $a_f(n)$ for f a cusp form $f(z)=\sum_{n=1}^{\infty} a_f(n)e(nz)$)

$a_f(n) \ll n^{k/2}$ (Lecture 6)

we have for A of $\det A = 1$ and $\operatorname{diag} A \equiv \operatorname{diag} A^{-1} \equiv 0\ (2)$

$$r(n,Q) = \frac{(2\pi)^{r/2}}{\zeta(r/2)\Gamma(r/2)} \sigma_{r/2-1}(n) + O(n^{r/4}).$$

This gets clumsy for general A.

The Circle Method after Kloosterman:

We study the Fourier coefficients directly:

$$r(n,Q) = \int_0^1 \Theta(z,A)\, e(-nz)\,dx, \text{ where } z = x+iy.$$

Goal for remainder of course:

Theorem Let Q be a positive-definite quadratic form in $r \geq 4$ variables with integer coefficients. Let $r/2 = k$.

For any $n > 0$ we have

$$r(n,Q) = \frac{(2\pi)^k\, n^{k-1}}{\zeta(k)\Gamma(k)(\det A)^{1/2}} \mathcal{S}(n,Q) + O_{Q,\varepsilon}\left(n^{k/2 - 1/4 + \varepsilon}\right)$$

where $\mathcal{S}(n,Q) = \sum_{c=1}^{\infty} \frac{g_c(n,c)}{c^r}$, and $g_c(n,Q) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \sum_{h \bmod c} e\left(\frac{d}{c}(Q(h)-n)\right)$.

Remarks: (1) The case $r=1$ is trivial,

The case $r=2$ is algebraic, eg

$r(n, x^2+y^2)$ is about ideals in $\mathbb{Z}[i]$

The case $r=3$ is hard, requires additional ideas...

(2) We still need to show the main term does not vanish. Later!
In fact, it will vanish for some (n, Q)!

(3) We don't need the full strength of the modularity of $\Theta(z, A)$ to prove this Theorem. Recall from the first Prop of Lecture 17:
(Last Thursday)

$$\sum_{\substack{m \in \mathbb{Z}^r \\ m \equiv h (c)}} e\left(\tfrac{1}{2} A[m] z\right) = (\det A)^{-1/2} c^{-r} \left(\frac{i}{z}\right)^{r/2} \sum_{m \in \mathbb{Z}^r} e\left(\frac{-\frac{1}{2} A^{-1}[m]}{c^2 z} - \frac{{}^{tr}hm}{c}\right)$$

Idea of proof of theorem:

We expect the integrand of
$$r(n, Q) = \int_0^1 \Theta(z, A) e(-nx) dx$$ to depend on ~~the~~ rational approximation to x with small denom.

We choose a decomposition of $(0,1]$ adapted to this idea.

Let $F_{\mathcal{C}} = \left\{ \frac{a}{c} : \underset{0 \le a < c}{(a,c) = 1}, \ 1 \le c \le \mathcal{C} \right\}$. $\mathcal{C} \in \mathbb{N}$.

This is called the Farey sequence of depth $\mathcal{C}$.

$F_{\mathcal{C}}$ comes with the natural order, eg

$$F_5 = \left\{ 0, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \right\}.$$

Let $\frac{a}{c} \in F_{\mathcal{C}}$, and write the adjacent elements of the Farey seq:

$$\frac{a'}{c'} < \frac{a}{c} < \frac{a''}{c''}$$

Exercise Given $\frac{a}{c} \in F_{\mathcal{C}}$, the adjacent c', c'' are the unique positive integers satisfying:

$$\begin{cases} \mathcal{C} - c < c' \leq \mathcal{C} & \text{and} \quad ac' \equiv 1\,(c) \\ \mathcal{C} - c < c'' \leq \mathcal{C} & \text{and} \quad ac'' \equiv -1\,(c) \end{cases}$$

and $a' = (c'a-1)c^{-1}$, $a'' = (ac''+1)c^{-1}$.

Between adjacent elements of $F_{\mathcal{C}}$ are the "mediants":

$$\frac{a'}{c'} < \frac{a+a'}{c+c'} < \frac{a}{c} < \frac{a+a''}{c+c''} < \frac{a''}{c''}$$

(Check this!) Note the mediants $\notin F_{\mathcal{C}}$, but $\in F_{\mathcal{C}+c}$.

Define the Farey segment around $\frac{a}{c}$:

$$\mathcal{M}\left(\frac{a}{c}\right) = \left(\frac{a+a'}{c+c'}, \frac{a+a''}{c+c''}\right] = \left(\frac{a}{c} - \frac{1}{c(c+c')}, \frac{a}{c} + \frac{1}{c(c+c'')}\right]$$

So: $$\left(\frac{-1}{\mathcal{C}+1}, \frac{\mathcal{C}}{\mathcal{C}+1}\right] = \bigsqcup_{\frac{a}{c} \in F_{\mathcal{C}}} \mathcal{M}\left(\frac{a}{c}\right) = \bigsqcup_{\substack{0 \leq a < c \leq \mathcal{C} \\ (a,c)=1}} \mathcal{M}\left(\frac{a}{c}\right).$$

Projecting $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$, this covers the whole interval $(0,1]$.

Let $f: \mathbb{R} \to \mathbb{C}$ be a periodic function of period 1, integrable on $[0,1]$.
We get an integration formula adapted to the arithmetic of these fractions:

$$\int_0^1 f(x)\,dx = \sum_{\substack{0 \leq a < c \leq \mathcal{C} \\ (a,c)=1}} \int_{\mathcal{M}(\frac{a}{c})} f(x)\,dx$$

$$= \sum_{0 \leq a < c \leq \mathcal{C}} \sum_{(a,c)=1} \int_{-\frac{1}{c(c+c')}}^{\frac{1}{c(c+c'')}} f\left(x - \frac{a}{c}\right) dx.$$

Let $d = c+c'$, resp. $c+c''$, so that $(d,c)=1$. (by the exercise).

We write (traditional notation in analytic number theory) $\bar{d} = d^{-1} \in (\mathbb{Z}/c)^{\times}$.

Thus:

$$\int_0^1 f(x)\,dx = \sum \sum_{\substack{1\le c\le \mathcal{C} < d \le c+\mathcal{C}\\ (d,c)=1}} \left(\int_{-\frac{1}{dc}}^{0} f\left(\frac{\bar{d}}{c}+x\right)dx + \int_{0}^{\frac{1}{dc}} f\left(-\frac{\bar{d}}{c}+x\right)dx \right).$$

Now let's assume f has real Fourier coefficients (i.e. $\overline{f(x)} = f(-x)$).

Lemma (Integration Formula)

$$\int_0^1 f(x)\,dx = 2\,\mathrm{Re}\left(\sum_{1\le c\le \mathcal{C}} \int_0^{\frac{1}{c\mathcal{C}}} \sum_{\substack{\mathcal{C}<d\le \mathcal{C}+c\\ cdx<1\\ (c,d)=1}} f\left(x-\frac{\bar{d}}{c}\right)dx \right)$$

Now we assume $r \geq 4$.

We write $z = x+iy$, and apply the integration formula to

$$f(x) = \Theta(z,A)\,e(-nx), \quad \text{with } y>0 \text{ to be chosen later.}$$

Thus:

$$r(n,Q) = 2\,\mathrm{Re} \sum_{1\le c\le \mathcal{C}} \int_0^{\frac{1}{c\mathcal{C}}} T(c,n,x)\,e(-nx)\,dx$$

where

$$T(c,n,x) = \sum_{\substack{\mathcal{C}<d\le c+\mathcal{C}\\ (c,d)=1\\ cdx<1}} e\left(n\frac{\bar{d}}{c}\right)\Theta\left(z-\frac{\bar{d}}{c},A\right).$$

We split the Θ-fcn into congruence Θ-fcns for residue classes $h \in \mathbb{Z}^r \bmod c$.

$$\Theta\left(z-\frac{\bar{d}}{c},A\right) = \sum_{h \bmod c} e\left(-\frac{\bar{d}}{c}\,\frac{1}{2}A[h]\right) \sum_{m\equiv h(c)} e\left(\frac{1}{2}A[m]z\right)$$

Apply the inversion formula:

$$\Theta\left(z-\frac{d}{c}, A\right) = (\det A)^{-1/2} c^{-r} \left(\frac{i}{z}\right)^{r/2} \sum_{m \in \mathbb{Z}^r} G_m\left(\frac{d}{c}\right) e\left(-\frac{\frac{1}{2}A^{-1}[m]}{c^2 z}\right)$$

where $$G_m\left(\frac{d}{c}\right) = \sum_{h \in \mathbb{Z}^r \bmod c} e\left(-\frac{d}{c}\left(A[h] + \operatorname{tr} hm\right)\right)$$

This is a multivariable Gauss sum.

We have:

$$T(c,n,x) = (\det A)^{-1/2} c^{-r} \left(\frac{i}{z}\right)^{r/2} T_m(c,n,x)\, e\left(-\frac{\frac{1}{2}A^{-1}[m]}{c^2 z}\right)$$

with $$T_m(c,n,x) = \sum_{\substack{\mathcal{C} < d \le \mathcal{C}+c \\ (c,d)=1 \\ cdx<1}} e\left(n\frac{d}{c}\right) G_m\left(-\frac{d}{c}\right).$$

We study these Gauss sums in more detail next class...