Classical Modular Forms — Lecture 2 — Sept 21, 2017

Recall from last time, for $(a,c)=1$ we defined the Gauss sum:

$$G(a,c) = \sum_{\alpha (c)} e\left(\frac{a\alpha^2}{c}\right) \in \mathbb{C}$$

$a \longmapsto G(a,c)$ is a function on $(\mathbb{Z}/c\mathbb{Z})^\times$

FACTS

① $\overline{G(a,c)} = \sum_{\alpha (c)} e\left(\frac{-a\alpha^2}{c}\right) = G_1(-a,c)$

② $G(d,4) = 2\sqrt{2i}\,\varepsilon_d^{-1}$

where $\varepsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 (4) \\ i & \text{if } d \equiv 3 (4) \end{cases}$,

③ If $(b,c)=1$ then $G(a,c) = G(ab^2,c)$

④ If $(c,d)=1$, $c,d>0$ c even then

$$\frac{G(-2c,d)}{d^{1/2}} = \frac{G(d,2c)}{2(ic)^{1/2}}$$

⑤ $$G(1,d) = d^{1/2}\varepsilon_d$$

(Exercise: Apply ①, ②, ③, ④.)

As another application of $\Theta(\gamma)$, and in order to state the transformation laws more cleanly, we study:

Def (Jacobi Symbol)

For $d>0$ odd, $(c,d)=1$ any integer

$$\left(\frac{c}{d}\right) = \frac{G(c,d)}{G(1,d)}.$$

This defines a function $(\mathbb{Z}/d)^\times \longrightarrow \mathbb{C}$.

In fact, by ③ $\left(\frac{c(c')^2}{d}\right) = \left(\frac{c}{d}\right)$ so defines a fcn on $(\mathbb{Z}/d)^\times/(\mathbb{Z}/d)^{\times 2}$

We compute $\left(\frac{c}{d}\right)$ through several reductions:

First, suppose $c>0$, c even. Then for any d_1 coprime to c,

$$\frac{G(-2c, dd_1^2)}{(dd_1^2)^{1/2}} \overset{④}{=} \frac{G(dd_1^2, 2c)}{2(ic)^{1/2}} \overset{③}{=} \frac{G(d, 2c)}{2(ic)^2} \overset{④}{=} \frac{G(-2c, d)}{d^{1/2}}.$$

Let $c' \equiv -2^{-1}c \pmod{dd_1^2}$, i.e. $\exists\, c'>0$ s.t. $c \equiv -2c' \pmod{dd_1^2}$
since dd_1^2 is odd

Then for any c with $(c, dd_1)=1$ we have

$$\frac{G(c, dd_1^2)}{(dd_1^2)^{1/2}} = \frac{G(c,d)}{d^{1/2}}, \text{ so } \left(\frac{c}{dd_1^2}\right) = \left(\frac{c}{d}\right) \text{ for } (c, dd_1)=1,\ d \text{ odd} >0.$$

~~Suffices to compute $\left(\frac{c}{d}\right)$ for d squarefree~~ d odd.

What if c is even or negative? Then we can compute by hand:

Exercise: $\left(\frac{1}{d}\right)=1$

$$\left(\frac{-1}{d}\right) = \varepsilon_d^{-2} = (-1)^{\frac{d-1}{2}} = \chi_4(d)$$

The unique non-trivial character $\chi_4 : (\mathbb{Z}/4)^\times \to \mathbb{C}^\times$.

$$\left(\frac{2}{d}\right) = e\left(-\frac{d-1}{8}\right)\varepsilon_d^{-1} = \begin{cases} 1 & d \equiv 1, 7 \ (8) \\ -1 & d \equiv 3, 5 \ (8) \end{cases} = (-1)^{\frac{d^2-1}{8}} = \chi_8(d)$$

χ_8 is the non-trivial homomorphism $\chi_8 : (\mathbb{Z}/8)^\times \to \{\pm 1\}$
distinct from χ_4 through the reduction $(\mathbb{Z}/8)^\times \to (\mathbb{Z}/4)^\times$.

THUS we have reduced the computation of $\left(\frac{c}{d}\right)$ to the case of d odd and square-free.

Chinese Remainder Theorem: Let d any positive integer, $d = d_1 d_2$ w/ $(d_1, d_2)=1$

then $(\mathbb{Z}/d_1)^\times \times (\mathbb{Z}/d_2)^\times \longrightarrow (\mathbb{Z}/d)^\times$ via $(x_1, x_2) \mapsto x_1 d_2 + d_1 x_2$ is an isom.

therefore we have $(x_1 d_2 + d_1 x_2)^2 \equiv x_1^2 d_2^2 + x_2^2 d_1^2 \ (d)$

So $G(c,d) = G(d_2 c, d_1)\, G(d_1 c, d_2)$ (*)

So more generally: $d = \prod_{p \mid d} p^{\alpha_p}$, and let $d_p = \frac{d}{p^{\alpha_p}}$.

Then $(d_p, p) = 1$ and

$$\boxed{G(c,d) = \prod_{p \mid d} G(d_p c, p^{\alpha_p})}$$

So, it suffices to compute $G(c,p)$ for p an odd prime.

The Legendre (Quadratic Residue) Symbol.

$(\mathbb{Z}/p)^{\times 2} \leq (\mathbb{Z}/p)^\times$ is a subgroup of index 2.

$R(p) = (\mathbb{Z}/p)^{\times 2} \leq \mathbb{Z}/p$, $NR(p) = R(p)^c$

Let b any $b \in NR(p)$. by fact ③:

$$G(a,p) = \begin{cases} G(1,p) & \text{if } a \in R(p) \\ G(b,p) & \text{if } a \in NR(p). \end{cases}$$

We have 2-to-1 maps

$(\mathbb{Z}/p)^\times \longrightarrow R(p)$ $\quad$ $(\mathbb{Z}/p)^\times \longrightarrow NR(p)$

$\alpha \longmapsto \alpha^2$ $\quad$ $\alpha \longmapsto b\alpha^2$

Thus $G(1,p) + G(b,p) = 2 + 2 \sum_{\alpha \in (\mathbb{Z}/p)^\times} e(\alpha/p) = 2 \sum_{\alpha (p)} e(\alpha/p) = 0$

So $G(b,p) = -G(1,p)$

$$\Longrightarrow \left(\frac{c}{p}\right) = \begin{cases} 1 & \text{if } c \in R(p) \\ -1 & \text{if } c \in NR(p). \end{cases}$$

Multiplication by $a_1 \in R(p)$ preserves $R(p)$ & $NR(p)$

multiplication by $a_2 \in NR(p)$ swaps $R(p)$ & $NR(p)$

So: $\left(\frac{\cdot}{p}\right) : (\mathbb{Z}/p)^\times \longrightarrow \pm\{1\}$ is a homomorphism called the Legendre symbol.

Thus, for any odd sqfree $d>0$

$$\left(\frac{c}{d}\right) = \prod_{p \mid d} \left(\frac{c}{p}\right).$$

Theorem For $d = \prod_{p \mid d} p^{\alpha_p}$ an odd, positive integer

Let $(c,d)=1$. Then

$$\left(\frac{c}{d}\right) = \prod_{p \mid d} \left(\frac{c}{p}\right)^{\alpha_p}, \text{ where } c \longmapsto \left(\frac{c}{p}\right) \text{ is the Legendre Symbol.}$$

So $\left(\frac{c}{d}\right)$ is a character mod d, i.e. $\left(\frac{\cdot}{d}\right): (\mathbb{Z}/d)^\times \to \{\pm 1\}$.

Theorem (Quadratic Reciprocity)

Given two coprime integers $c,d>1$, odd we have

$$\left(\frac{c}{d}\right)\left(\frac{d}{c}\right) = (-1)^{\frac{d-1}{2}\cdot\frac{c-1}{2}}$$

Proof Let $c>0$ $(c,2d)=1$ then

$$\left(\frac{-4c}{d}\right) = \frac{G(-4c,d)}{G(1,d)} \overset{(4)}{=} \frac{d^{1/2}\, G(d,4c)}{2(2ic)^{1/2}} \frac{1}{G(1,d)} \overset{(5)}{=} \frac{d^{1/2} G(d,4c)}{2(2ic)^{1/2}} \frac{1}{d^{1/2}\varepsilon_d}$$

$$= \frac{G(4d,c)\, G(cd,4)}{2(2ic)^{1/2}\varepsilon_d} \overset{(2)(3)}{=} \frac{2\sqrt{2i}\,\varepsilon_{cd}^{-1}\, G(d,c)}{2(2ic)^{1/2}\varepsilon_d} \overset{(5)}{=} \frac{\varepsilon_{cd}^{-1}}{c^{1/2}\varepsilon_d} G(d,c) \frac{c^{1/2}\varepsilon_c}{G(1,c)}$$

$$= \left(\frac{d}{c}\right) \frac{\varepsilon_{cd}^{-1}\varepsilon_c}{\varepsilon_d}$$

Factorization of Gauss Sums

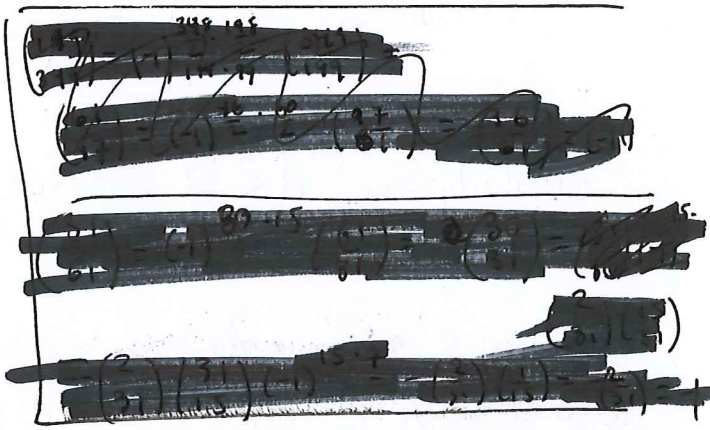
Since $\left(\frac{-1}{d}\right) = \varepsilon_d^{-1} \Rightarrow \left(\frac{c}{d}\right) = \frac{\varepsilon_d \varepsilon_c}{\varepsilon_{cd}} \left(\frac{d}{c}\right) = (-1)^{\frac{c-1}{2}\frac{d-1}{2}} \left(\frac{d}{c}\right)$ Q.E.D.

Remark: If $(c,d)=(p,q)$ distinct primes, then the exercise & Quadratic reciprocity give a rapid way to compute whether p is a $\square$ mod q.

Remark: Quadratic Reciprocity was first discovered by Gauss who gave several proofs. (5/6)

The above proof is due to Hecke, and is essentially analytic in nature.

For the following, we extend the definition of $\left(\frac{c}{d}\right)$ to d odd and arbitrary $c \in \mathbb{Z}$ by:



① $\left(\frac{c}{d}\right) = 0$ if $(c,d) \neq 1$

② If $c \neq 0$ $\left(\frac{c}{-d}\right) = \frac{c}{|c|}\left(\frac{c}{d}\right)$

③ $\left(\frac{0}{d}\right) = \begin{cases} 1 & \text{if } d = \pm 1 \\ 0 & \text{else.} \end{cases}$

Theorem (Summary) Let $\left(\frac{c}{d}\right)$ the extended Jacobi symbol, $d \equiv 1 (2)$.

① If $d = \prod_p p^{\alpha_p} > 0$, then $c \mapsto \left(\frac{c}{d}\right)$ is a character of $\left(\mathbb{Z}/d\right)^\times$

and $\left(\frac{c}{d}\right) = \prod_{p \mid d} \left(\frac{c}{p}\right)^{\alpha_p}$, where $\left(\frac{c}{p}\right)$ is the Legendre symbol.

② For $c \neq 0$ the map $d \to \left(\frac{c}{d}\right)$ defines a character of $\left(\mathbb{Z}/4|c|\right)^\times$ which is even if $c > 0$ and odd if $c < 0$, i.e.

$$\left(\frac{c}{-d}\right) = \left(\frac{c}{d}\right) \text{ if } c > 0, \quad \left(\frac{c}{-d}\right) = -\left(\frac{c}{d}\right) \text{ if } c < 0.$$

③ In particular, $\left(\frac{-1}{d}\right) = \chi_4(d) = (-1)^{\frac{d-1}{2}}$, $\left(\frac{2}{d}\right) = \chi_8(d) = (-1)^{\frac{d^2-1}{8}}$

④ For c odd $\left(\frac{c}{d}\right) = \begin{cases} \chi_4(d)^{\frac{c-1}{2}} \left(\frac{d}{c}\right) & \text{if } c > 0 \\ \chi_4(d)^{\frac{|c|+1}{2}} \left(\frac{d}{|c|}\right) & \text{if } c < 0. \end{cases}$

Theorem For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(2)$ we have

$$\hat{\Theta}(\gamma z) = \left(\frac{2c}{d}\right) \varepsilon_d^{-1} (cz+d)^{1/2} \hat{\Theta}(z)$$

Proof Let $\gamma \in \Gamma(2)$. If $c=0$, $d=\pm 1$, $\gamma = \pm T^b$.

Then $\widehat{\Theta}(\gamma z) = \widehat{\Theta}(z) = \left(\frac{0}{d}\right) \varepsilon_d^{-1} d^{1/2} \widehat{\Theta}(z)$ by the defn of the extended Jacobi symbol (since $d=\pm 1$).

If $c \neq 0$, and $d>0$, then from the computation last week:

$$\widetilde{\Theta}(\gamma z) = \left(\frac{-c/2}{d}\right) \frac{G(1,d)}{d^{1/2}} (cz+d)^{1/2} \widetilde{\Theta}(z) = \left(\frac{2c}{d}\right) (-1)^{\frac{d-1}{2}} \varepsilon_d (cz+d)^{1/2} \widetilde{\Theta}(z)$$

$$= \left(\frac{2c}{d}\right) \varepsilon_d^{-1} (cz+d)^{1/2} \widetilde{\Theta}(z) \quad \text{since } \varepsilon_d^2 = (-1)^{\frac{d-1}{2}}.$$

Similarly for $d<0$ since $-Id$ acts trivially

Q.E.D.

Perhaps even cleaner / more standard:

Let $\Theta(z) := \widehat{\Theta}(2z) = \sum_{n \in \mathbb{Z}} e(n^2 z)$

Note $\forall \gamma \in GL_2^+(\mathbb{R})$ $\Theta(\gamma z) = \widetilde{\Theta}\left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \gamma z\right) = \widetilde{\Theta}\left(\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} 2z\right)$

and $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \Gamma(2) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \Gamma_0(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} : c \equiv 0 (4) \right\}$

COROLLARY

$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$

$\Theta(\gamma z) = j_{1/2}(\gamma, z)\, \Theta(z)$, where

$j_{1/2}(\gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} j(\gamma, z)^{1/2}$ and $j(\gamma, z) = cz+d$.

The functions $j_{1/2}(\gamma, z)$ and $j(\gamma, z)$ are called multiplier systems.

Note since $\Theta(\gamma\gamma' z) = \Theta(\gamma(\gamma' z))$, we have $\forall \gamma, \gamma' \in \Gamma_0(4)$

$j_{1/2}(\gamma\gamma', z) = j_{1/2}(\gamma, \gamma' z)\, j_{1/2}(\gamma', z)$ a cocycle relation.

and similarly $j(\gamma\gamma', z) = j(\gamma, \gamma' z) j(\gamma', z)$ $\forall \gamma, \gamma' \in GL_2^+(\mathbb{R})$.