Classical Modular Forms — Lecture 20 — 14.12.2017

Theorem Let Q be a positive-definite quadratic form in $r \geq 4$ variables with integer coeffs. Let $r/2 = k$.

For any $n \geq 1$ we have $\forall \varepsilon > 0$

$$r(n,Q) = \frac{(2\pi)^k n^{k-1}}{\Gamma(k)\zeta(k)(\det A)^{1/2}} \mathcal{S}(n,Q) + O_{Q,\varepsilon}\left(n^{k/2 - 1/4 + \varepsilon}\right)$$

where $\mathcal{S}(n,Q) = \sum_{c=1}^{\infty} \frac{g_c(n,Q)}{c^r}$

$$g_c(n,Q) = \sum_{\substack{d \bmod c \\ (d,c)=1}} \sum_{h \in (\mathbb{Z}/c)^r} e\left(\frac{d}{c}(Q(h) - n)\right)$$

Strategy: apply circle method to $\Theta(z,A)$.

Lemma Let $f : \mathbb{R} \to \mathbb{C}$ be periodic with period 1, integrable on $[0,1)$, with real Fourier coefficients $\left(\Leftrightarrow \overline{f(x)} = f(-x)\right)$, $\mathcal{C} \geq 1$.

then
$$\int_0^1 f(x)\,dx = 2 \operatorname{Re}\left(\sum_{1 \leq c \leq \mathcal{C}} \int_0^{\frac{1}{c\mathcal{C}}} \sum_{\substack{\mathcal{C} < d \leq \mathcal{C}+c \\ cdx < 1 \\ (c,d)=1}} f\left(x - \frac{\bar{d}}{c}\right) dx \right)$$

Here $\bar{d}$ denotes inverse of d mod c.

Recall, we used Farey fractions to prove this.

Now, assume $r \geq 4$

Write $z = x + iy$, apply integration formula to $f(x) = \Theta(z,A)e(-nz)$

Thus: $r(n,Q) = \int_0^1 \Theta(z,A)e(-nz)\,dx = 2\operatorname{Re} \sum_{1 \leq c \leq \mathcal{C}} \int_0^{\frac{1}{c\mathcal{C}}} T(n,c,x)e(-nz)\,dx$

where $T(c,n,x) = \sum_{\substack{\mathcal{C} < d \leq \mathcal{C}+c \\ (c,d)=1 \\ cdx<1}} e\left(n\frac{d}{c}\right)\Theta\left(z-\frac{d}{c},A\right)$

Next we split the Θ-fcn over residue classes into congruence Θ-fcns:

$$\Theta\left(z-\frac{d}{c},A\right) = \sum_{h\in(\mathbb{Z}/c)^r} e\left(-\frac{d}{c}\cdot\frac{1}{2}A[h]\right)\sum_{m\equiv h(c)} e\left(\frac{1}{2}A[m]z\right)$$

Recall the inversion formula (first Prop of Lecture 17):

$$\sum_{\substack{m\in\mathbb{Z}^r \\ m\equiv h(c)}} e\left(\frac{1}{2}A[m]z\right) = (\det A)^{-1/2}c^{-r}\left(\frac{i}{z}\right)^{r/2}\sum_{m\in\mathbb{Z}^r} e\left(-\frac{\frac{1}{2}A^{-1}[m]}{c^2z} - \frac{{}^t h m}{c}\right).$$

Therefore

$$\Theta\left(z-\frac{d}{c},A\right) = (\det A)^{-1/2}c^{-r}\left(\frac{i}{z}\right)^{r/2}\sum_{m\in\mathbb{Z}^r} G_m\left(\frac{-d}{c}\right) e\left(\frac{-\frac{1}{2}A^{-1}[m]}{c^2z}\right)$$

with $G_m\left(\frac{d}{c}\right) = \sum_{h\in(\mathbb{Z}/c)^r} e\left(\frac{d}{c}\left(\frac{1}{2}A[h] + {}^t h m\right)\right)$

(where we've changed h to $dh \bmod c$).

This sum is a multivariable Gauss sum.

We find $T(c,n,x) = (\det A)^{-1/2}c^{-r}\left(\frac{i}{z}\right)^{r/2} T_m(c,n,x)\, e\left(\frac{-\frac{1}{2}A^{-1}[m]}{c^2z}\right)$

with $T_m(c,n,x) = \sum_{\substack{\mathcal{C} < d \leq \mathcal{C}+c \\ (c,d)=1 \\ cdx<1}} e\left(n\frac{d}{c}\right)G_m\left(-\frac{d}{c}\right)$

The goal is to estimate T_m.

First we estimate G_m.

There are two cases: $(c, 2\det A) = 1$ or not.

The general case is simpler:

Lemma Let $(c,d)=1$ and $m \in \mathbb{Z}^r$. We have

$$|G_m(\tfrac{d}{c})| \le (\det A)^{1/2} c^{r/2} \ll_A c^{r/2}$$

Proof

$$|G_m(\tfrac{d}{c})|^2 = \sum_{x,y \in (\mathbb{Z}/c)^r} e\left(\frac{d}{c}\left(\tfrac{1}{2}A[x] - \tfrac{1}{2}A[y] + {}^{tr}(x-y)\cdot m\right)\right)$$

Let $x-y=z$

$$= \sum_{y,z \in (\mathbb{Z}/c)^r} e\left(\frac{d}{c}\left(\tfrac{1}{2}A[z] + {}^{tr}y A z + {}^{tr}z.m\right)\right)$$

orthogonality of chars over the y-sum, trivial over z sum:

$$\le c^r |\{z \bmod c : Az \equiv 0 \,(c)\}|$$

$$= c^r |H| = c^r \det A.$$

Lemma Let $(c, 2(\det A)\cdot d)=1$, $m \in \mathbb{Z}^r$. We have

$$G_m(\tfrac{d}{c}) = \left(\frac{\det A}{c}\right)\left(\varepsilon_c\left(\frac{2d}{c}\right)\sqrt{c}\right)^r e\left(-\frac{\bar{d}}{c}\tfrac{1}{2}A^{-1}[m]\right).$$

Note: this also finishes the proof of modularity of $\Theta(z,A)$.

Proof The idea is to Diagonalize and complete the square to reduce to the 1-dimensional case. You showed in exercise 1.12 if $d>0$, $(c,2a)=1$ that

$$G(a,c) = \sum_{y \bmod c} e\left(\frac{ay^2}{c}\right) = \varepsilon_c\left(\frac{a}{c}\right)\sqrt{c}.$$

Fact from linear algebra: If k is any field of char $k \neq 2$, then every symmetric matrix is equiv. to a diagonal one in the sense that there exist U invertible such that

$${}^{tr}UAU = D \text{ with } D = \mathrm{diag}(d_1, \dots, d_r).$$

Factoring $c = p_1^{e_1} \cdots p_n^{e_n}$ with $p_i \neq 2$, we can reduce $A \bmod p_i$ produce U and $D \bmod p_1^{e_1}$, and then by Chinese remainder theorem lift to matrices $V \in M_r(\mathbb{Z})$ such that

$${}^{tr}VAV \equiv B \mod c \quad \text{with } B = \mathrm{diag}(b_1, \dots, b_r)$$

$b_i \in \mathbb{Z}$.

$$G_m\left(\frac{d}{c}\right) = \sum_{h \in (\mathbb{Z}/c)^r} e\left(\frac{d}{c}\left(\frac{1}{2}A[h] + {}^{tr}hm\right)\right)$$

Let $h = Vy \mod c$

So $A[h] \equiv B[y] \equiv \sum_{v=1}^{r} b_v y_v^2 \mod c$.

Let ${}^{tr}Vm = [d_1, \dots, d_r]$, so ${}^{tr}hm = {}^{tr}y\,{}^{tr}Vm = \sum_{v=1}^{r} d_v y_v$

Thus $\frac{1}{2}A[h] + {}^{tr}hm = \frac{1}{2}\sum_{v=1}^{r}(b_v y_v^2 + 2d_v y_v)$

$$= \frac{1}{2}\sum_{v=1}^{r} b_v (y_v + b_v^{-1} d_v)^2 - \frac{1}{2}\sum_{v=1}^{r} b_v^{-1} d_v^2$$

(Complete the square) (here b_v^{-1} is inverse mod c)

$(c, \det A) = 1$, so each $(b_v, c) = 1$.

and $\sum_{v=1}^{r} b_v^{-1} d_v^2 = B^{-1}[{}^{tr}Vm] = V^{-1}A^{-1}({}^{tr}V^{-1})[{}^{tr}Vm] = A^{-1}[m] \mod c$.

So $$G_m\left(\frac{d}{c}\right) = \sum_{y \bmod c} e\left(\frac{d}{c}\left(\frac{1}{2}\sum_{v=1}^{r} b_v (y_v + b_v^{-1} d_v)^2 - \frac{1}{2}A^{-1}[m]\right)\right)$$

$$= e\left(-\frac{1}{2}A^{-1}[m]\frac{d}{c}\right)\prod_{v=1}^{r}\sum_{y_v \bmod c} e\left(\frac{d}{c}\frac{1}{2} b_v (y_v + b_v^{-1} d_v)^2\right)$$

$$= e\left(-\tfrac{1}{2}A^{-1}[m]\tfrac{d}{c}\right)\prod_{v=1}^{r} G(2db_v, c)$$

$$= e\left(-\tfrac{1}{2}A^{-1}[m]\tfrac{d}{c}\right)\prod_{v=1}^{r} \varepsilon_c\left(\frac{2db_v}{c}\right)\sqrt{c}$$

Since $b_1 \cdots b_r = \det B = (\det V)^2 \det A$

$$= \left(\varepsilon_c\sqrt{c}\left(\frac{2d}{c}\right)\right)^r \left(\frac{\det A}{c}\right) e\left(-\tfrac{1}{2}A^{-1}[m]\tfrac{d}{c}\right).$$

Now we complete

$$T_m(c,n,x) = \sum_{\substack{\mathscr{C} < d \le \mathscr{C}+c \\ (c,d)=1 \\ cdx<1}} e\left(n\frac{d}{c}\right) G_m\left(\frac{-d}{c}\right)$$ by a separation of variables trick:

$$= \sum_{\substack{d(c) \\ (d,c)=1}} e\left(\frac{nd}{c}\right) G_m\left(\frac{-d}{c}\right) \sum_{\mathscr{C}<b\le \min(c+\mathscr{C}, \frac{1}{cx})} \delta_{b=d}$$

use $\delta_{b=d} = \frac{1}{c}\sum_{\ell \bmod c} e\left(\frac{(-b+d)\ell}{c}\right)$

Thus: $T_m(c,n,x) = \sum_{\ell \bmod c} \gamma(\ell)\, K(\ell,m,n,c)$ with

$$K(\ell,m,n,c) = \sum_{\substack{d(c) \\ (d,c)=1}} e\left(\frac{\ell d + nd}{c}\right) G_m\left(\frac{-d}{c}\right)$$

$$\gamma(\ell) = \frac{1}{c}\sum_{\mathscr{C}<b\le \min(c+\mathscr{C}, \frac{1}{cx})} e\left(\frac{-b\ell}{c}\right)$$

If $|\ell| \le c/2$ then I claim $\gamma(\ell) \ll \frac{1}{1+|\ell|}$.

If $l = 0 \to$ take $\mathscr{C} + c$ in min $\Rightarrow \leq 1$.

If $l \neq 0$ use Geometric Series: if $|l| \leq c/2$

$$\gamma(l) = \frac{1}{c} \frac{e(*) - e(*)}{1 - e(-l/c)} \ll \frac{1}{|l|}, \text{ so } \gamma(l) \ll \frac{1}{1+|l|}.$$

$$e(-l/c) = 1 - \frac{2\pi i l}{c} + O\left(\frac{l^2}{c^2}\right)$$

Next we factor $K(l,m,n,c)$.

Note if $c = c_0 c_1$ with $(c_0, c_1) = 1$, then

$\bar{c}_0 c_0 + \bar{c}_1 c_1 \equiv 1 \mod c_0 c_1$, where $\bar{c}_0$ is any lift of $c_0^{-1} \bmod c_1$ and $\bar{c}_1$ is any lift of $c_1^{-1} \bmod c_0$.

Therefore

$$e\left(\frac{x}{c_0 c_1}\right) = e\left(\frac{c_0 \bar{c}_0 + c_1 \bar{c}_1}{c_1 c_0} x\right) = e\left(\frac{\bar{c}_0 x}{c_1}\right) e\left(\frac{\bar{c}_1 x}{c_0}\right).$$

So we factor

$$K(l,m,n,c) = K^{(c_0)}(l,m,n,c_1) K^{(c_1)}(l,m,n,c_0)$$

where in $K^{(c_0)}(l,m,n,c_1)$ we replace every instance of $e\left(\frac{\circ}{c}\right)$ by $e\left(\frac{\bar{c}_0 \bullet}{c_1}\right)$, and similarly for $K^{(c_1)}(l,m,n,c_0)$.

Let $c_0 \mid 2 \det A$, and c_1 such that $(c_1, 2\det A) = 1$.

By the first lemma

$$K^{(c_1)}(l,m,n,c_0) \ll_Q c_0^{r/2+1} \qquad (\text{summing trivially over } d)$$

By the second lemma,

$$K^{(c_0)}(l,m,n,c_1) = \left(\frac{\det A}{c}\right)\left(\varepsilon_{c_1}\left(\frac{2d}{c_1}\right)\sqrt{c_1}\right)^r S_r\left(\bar{c}_0 l, \bar{c}_0 \left(n + \tfrac{1}{2} A^{-1}[m]\right)\right)$$

where $S_r(x,y) = \sum_{d \bmod c_1} \left(\frac{d}{c_1}\right)^r e\left(\frac{xd + y\bar{d}}{c_1}\right)$ is a Kloosterman Sum

Fact (Weil bound)

$$|S_r(x,y)| \le (x,y,c)^{1/2} c^{1/2} \tau(c)$$

The proof uses algebraic geometry and is beyond the scope of the course.

Thus $K^{(c_0)}(\ell,m,n,c_1) \ll (\ell, n+\frac{1}{2}A^{-1}[m], c_1)^{1/2} c_1^{\frac{r+1}{2}} \tau(c_1)$

Collecting these results we have shown:

Lemma

$$T_m(c,n,x) \ll_Q \left(\tfrac{1}{2}A^{-1}[m]+n, \ell, c_1\right)^{1/2} c_0^{1/2} c^{\frac{r+1}{2}} \tau(c) \log 2c.$$

We apply this bound to

$$T(c,n;x) = (\det A)^{-1/2} c^{-r} \left(\frac{i}{z}\right)^k \sum_{m\in\mathbb{Z}^r} T_m(c,n,x)\, e\left(\frac{-\frac{1}{2}A^{-1}[m]}{c^2 z}\right)$$

for all m except $m=0$.

$$T_0(c,n,x) = T(c,n) = \sum_{(d,c)=1} e\left(\frac{\bar{d}n}{c}\right) \sum_{h\in(\mathbb{Z}/c)^r} e\left(\frac{-d}{c}\frac{1}{2}A[h]\right)$$

Thus $T(c,n;x) = (\det A)^{-1/2} c^{-r} \left(\frac{i}{z}\right)^k T(n,c)$

$$+ O\left((c_0 c)^{1/2}\tau(c)\log 2c\, (c|z|)^{-k} \sum_{m\in\mathbb{Z}^r}^{b} \left(n+\tfrac{1}{2}A^{-1}[m], c_1\right)^{1/2} \exp\left(-\frac{\pi y A^{-1}[m]}{c^2|z|^2}\right)\right)$$

where b means $m=0$ is excluded from the sum if $0<x<\frac{1}{c(c+\gamma)}$.