Recall our goal is an asymptotic formula for

$$r(n,Q) = \# \{ x \in \mathbb{Z}^r : n = Q(x) \}.$$

We use the circle method: For $\mathcal{C} \geq 1$ we showed that

$$r(n,Q) = 2\,\mathrm{Re} \sum_{1 \leq c \leq \mathcal{C}} \int_0^{\frac{1}{c\mathcal{C}}} T(c,n,x)\, e(-nz)\,dx$$

where $T(c,n,x) = (\det A)^{-1} c^{-r} \left(\frac{i}{z}\right)^{r/2} \sum_{m \in \mathbb{Z}^r} T_m(c,n,x)\, e\left(\frac{-\frac{1}{2}A^{-1}[m]}{c^2 z}\right)$

write $Q^*(m) = \frac{1}{2}A^{-1}[m]$.

and $T_m(c,n,x) = \sum_{\substack{\mathcal{C} < d \leq \mathcal{C}+c \\ (c,d)=1 \\ cdx<1}} e\left(\frac{nd}{c}\right) G_m\left(\frac{-d}{c}\right)$

where G_m is a ~~multivariable~~ r-variable Gauss sum that we studied in detail.

We proved two Lemmas:

1. If $(c,d)=1$ $G_m\left(\frac{d}{c}\right) \ll c^{r/2}$
2. If $(c, 2d\det A)=1$ $G_m\left(\frac{d}{c}\right) = \left(\frac{\det A}{c}\right)\left(\varepsilon_c\left(\frac{2d}{c}\right)\sqrt{c}\right)^r e\left(-\frac{\bar{d}}{c} Q^*(m)\right).$

We completed $T_m(c,n,x) = \sum_{l \bmod c} \gamma(l)\, K(l,m,n,c)$

$\gamma(l) \ll \frac{1}{|l|+1}$ "analytic"

$K(l,m,n,c) = \sum_{\substack{d(c) \\ (d,c)=1}} e\left(\frac{ld + \bar{d}n}{c}\right) G_m\left(\frac{-d}{c}\right)$ "algebraic"

We showed that if $c = c_0 c_1$ w/ $(c_0, c_1) = 1$, and $c_0 \mid 2\det A$
$(c_1, 2\det A) = 1$

That $K(l, m, n, c) = K^{(c_1)}(l, m, n, c_0)\, K^{(c_0)}(l, m, n, c_1)$

where $K^{(c_0)}(\cdots, c_1)$ is defined as K but with $e(\frac{\cdot}{c})$ replaced by $e\left(\frac{\bar{c}_0 \cdot}{c_1}\right)$.

$\bar{c}_0$ is c_0 inverse mod c_1.

We showed: $K^{(c_1)}(l, m, n, c_0) \ll_Q c_0^{r/2+1}$ using ①

We also have

$$K^{(c_0)}(l, m, n, c_1) = \left(\frac{\det A}{c}\right)\left(\varepsilon_{c_1}\left(\frac{2d}{c_1}\right)\sqrt{c_1}\right)^r S_r(\bar{c}_0 l, \bar{c}_0 (n + Q^*(m)))$$

where $S_r(x, y) = \sum_{d \bmod c_1} \left(\frac{d}{c}\right)^r e\left(\frac{xd + y\bar{d}}{c_1}\right)$ by ② is a <u>Kloosterman sum</u>

<u>Fact</u> (Weil bound)

$$|S_r(x, y)| \le (x, y, c)^{1/2} c^{1/2} \sum_{d \mid c} 1 .$$

Not'n: $\tau(c) = \sum_{d \mid c} 1$.

The proof requires algebraic Geometry, and is beyond the scope of the course.

<u>Thus</u>: $K^{(c_0)}(l, m, n, c) \ll (l, n + Q^*(m), c_1)^{1/2} c_1^{\frac{r+1}{2}} \tau(c_1)$

<u>Lemma</u>

$$T_m(c, n, x) \ll_Q \left(Q^*(m) + n, l, c_1\right)^{1/2} c_0^{1/2} c^{\frac{r+1}{2}} \tau(c) \log 2c$$

We apply the lemma to the formulas for T above, for all m except $m = 0$ if x is in the range $0 < x < \frac{1}{c(c+\ell)}$.

For these n, x we have

$$T_0(c,n,x) = T(c,n) = \sum_{(d,c)=1} e\left(\frac{dn}{c}\right) \sum_{h\in(\mathbb{Z}/c)^r} e\left(\frac{-d}{c}\, Q(h)\right).$$

Thus: $$T(c,n,x) = (\det A)^{-1/2}\, c^{-r} \left(\frac{i}{z}\right)^k T(n,c)\, \delta_{0<x<\frac{1}{c(c+\mathscr{C})}}$$

$$+\, O\left((c_0 c)^{1/2}\, \tau(c) \log^2 c\, (c|z|)^{-k} \sum^{\flat}_{m\in\mathbb{Z}^r} (n+Q^*(m), c_1)^{1/2} \exp\left(\frac{-2\pi y\, Q^*(m)}{c^2|z|^2}\right)\right)$$

Where $\flat$ means $m=0$ is excluded from the sum if $0<x<\frac{1}{c(c+\mathscr{C})}$.

We estimate the sum in the error term by:

$$= \sum_{l\geq 0} \underbrace{(n + l/N,\, c_1)^{1/2}}\, (l+1)^{-2} \sum^{\flat}_{\substack{m\in\mathbb{Z}^r \\ l = NQ^*(m)}} (1+NQ^*(m))^2 \exp\left(\frac{-2\pi y\, Q^*(m)}{c^2|z|^2}\right)$$

since $(c_1, N)=1$

we replace this with $(Nn+l, c_1)^{1/2}$

here N is such that NA^{-1} has $\mathbb{Z}$ entries.

$$\leq \sum_{l\geq 0} (Nn+l, c_1)^{1/2} (l+1)^{-2} \sum^{\flat}_{m\in\mathbb{Z}^r} (1+NQ^*(m))^2 \exp\left(\frac{-2\pi y\, Q^*(m)}{c^2|z|^2}\right).$$

Now we make some choices. Recall we were free to choose $\mathscr{C}, y$. We take $\mathscr{C} = n^{1/2}$, $y = \mathscr{C}^{-2} = n^{-1}$

$$\Rightarrow\ c|z|y^{-1/2} \leq cy^{-1/2}(x+y) < (\mathscr{C}^{-1} + \mathscr{C}y)\, y^{-1/2} \quad \text{since } c\leq\mathscr{C},\ x<(\mathscr{C}c)^{-1},$$

$$\leq 2 \quad \text{since } 1\leq\mathscr{C}$$

If $(c(c+\mathscr{C}))^{-1} < x < (c\mathscr{C})^{-1}$ then we have

$$c|z|y^{-1/2} \geq cxy^{-1/2} > (c+\mathscr{C})^{-1} y^{-1/2} \geq (2\mathscr{C})^{-1} y^{-1/2} = 1/2 \quad \text{since } c\leq\mathscr{C}$$

since $\overset{*}{Q}(m) \gg \|m\|^2$ (4/7)

$$\sum_{m\in\mathbb{Z}^r}^{b} (1+NQ^*(m))^2 \exp\left(\frac{-2\pi y Q^*(m)}{c^2|z|^2}\right) \ll \delta_{(c(c+\mathscr{C}))^{-1} \leq x} + \sum_{m\neq 0} (1+NQ^*(m))^2 \left(\frac{y\|m\|^2}{c^2|z|^2}\right)^{-K}$$

we have $\delta_{(c(c+\mathscr{C}))^{-1}<x} \ll_K \left(\frac{c|z|}{y^{1/2}}\right)^K \quad \forall K \geq 0$

since if $x \leq c(c+\mathscr{C})^{-1}$ this term is 0, so the estimate is true

if $(c(c+\mathscr{C}))^{-1} < x$ then $1 \ll_K (1/2)^{K'}$ is also true.

$\ll 1.$

we also have the second sum is $\ll_{K,Q} \left(\frac{c|z|}{y^{1/2}}\right)^K$.

Thus, taking $K=k$ we have

$$T(c,n,x) = (\det A)^{-1/2} c^{-r} \left(\frac{i}{z}\right)^k T(c,n) + O\left(\xi(c_1)(c_0 c)^{1/2} \tau(c) \log 2c \; n^{k/2}\right)$$

where $\xi(c_1) = \sum_{l\geq 0} (c_1, l+nN)^{1/2} (l+1)^{-2}$

$$\Rightarrow r(n,Q) = (\det A)^{-1/2} \sum_{1\leq c\leq \mathscr{C}} c^{-r} T(c,n) \int_{-\frac{1}{c(c+\mathscr{C})}}^{\frac{1}{c(c+\mathscr{C})}} \left(\frac{i}{z}\right)^k e(-nz)\,dx.$$

$$+ O\left(n^{k/2} \mathscr{C}^{-1} \sum_{c\leq \mathscr{C}} \xi(c_1) \left(\frac{c_0}{c}\right)^{1/2} \tau(c) \log 2c\right)$$

The error term is bounded by

$$\ll_\varepsilon n^{k/2} \mathscr{C}^{\varepsilon-1} \sum_{c\leq \mathscr{C}} \xi(c_1) c_1^{-1/2} \ll n^{k/2} \mathscr{C}^{\varepsilon-1/2} \ll n^{\frac{k}{2}-\frac{1}{4}+\varepsilon}.$$

The integral is: $\int_{-\infty}^{\infty} \left(\frac{i}{z}\right)^k e(-nz)\,dx + O\left((c\mathscr{C})^{k-1}\right)$

Fact / Exercise
$$\int_{-\infty}^{\infty} \left(\frac{i}{z}\right)^k e(-nz)\,dx = \frac{(2\pi)^k n^{k-1}}{\Gamma(k)}$$

We have to estimate
$$\sum_{c \le \mathscr{C}} c^{-k/2} |T(c,n)| \, O((c\mathscr{C})^{k-1})$$
$$\hookrightarrow \ll (n, c_1)^{1/2} c_0^{1/2} c^{\frac{r+1}{2}} \tau(c) \log 2c$$

$$\text{So: } \ll \mathscr{C}^{k-1} \sum_{c \le \mathscr{C}} c^{r/2-1} c^{\frac{r+1}{2}} c^{-r} c_0^{1/2+\varepsilon} \ll \mathscr{C}^{k-1/2+\varepsilon} = n^{k/2-1/4+\varepsilon}.$$

Matches other error term.

Thus:
$$r(n,Q) = \frac{(2\pi)^k n^{k-1}}{\Gamma(k)(\det A)^{1/2}} \sum_{c \le \mathscr{C}} \frac{T(c,n)}{c^r} + O(n^{k/2-1/4+\varepsilon}).$$

Extend the sum to all $c \in \mathbb{N}$: Neglidible error.

Note $T(c,n) = g_c(n,Q)$ introduced last week by the transformations $h \to dh$ and $d \to -d$.

We defined $\displaystyle \mathfrak{S}(n,Q) = \sum_{c \ge 1} \frac{g_c(n,Q)}{c^r}$, $\displaystyle g_c(n,Q) = \sum_{\substack{d(c) \\ (d,c)=1}} \sum_{h(c)} e\left(\tfrac{d}{c}(Q(h)-n)\right)$

General philosophy:

$\mathfrak{S}(n,Q) = \prod_p \delta_p(n,Q)$, where δ_p are "local densities" of p-adic solutions to $Q(x)=n$.

$\displaystyle \delta_\infty(n,Q) = \frac{(2\pi)^k n^{k-1}}{\Gamma(k)\det A)^{1/2}}$ is density of real solutions to $Q(x)=n$.

We compute these δ_p for $p \nmid 2\det A$.

Note the sum over d is what is known as a Ramanujan Sum:

By computation on prime powers we can prove

$$R(b,c) := \sum_{\substack{d(c)\\(d,c)=1}} e\left(\frac{bd}{c}\right) = \sum_{q\mid(b,c)} \mu\left(\frac{c}{q}\right) q \qquad \text{multiplicative}$$

where $\mu(n) = \begin{cases} 1 & n \text{ sq free \& even \# of prime factors} \\ -1 & n \text{ `` `` odd \# `` ``} \\ 0 & n \text{ is \underline{not} sq free} \end{cases}$

Thus:

$$g_c(n,Q) = \sum_{q\mid c} \mu\left(\frac{c}{q}\right) q \,\#\{h \bmod c : Q(h) \equiv n \ (q)\}.$$

and

$$\sum_{c\mid s} \frac{g_c(n,Q)}{c^r} = s^{1-r} \#\{m \ (\bmod s) : Q(m) = n \ (\bmod s)\} = \prod_{p\mid s} \delta_p(n,Q) \quad \text{if } s \text{ suff large.}$$

On the other hand: via the Gauss sum formula (Lemma 2) if $(c, 2\det A) = 1$

$$G_0\left(\frac{d}{c}\right) = \sum_{h(c)} e\left(\frac{d}{c} Q(h)\right) = \left(\frac{\det A}{c}\right)\left(\varepsilon_c\left(\frac{2d}{c}\right)\sqrt{c}\right)^r$$

$$\Rightarrow g_c(n,Q) = \left(\frac{\det A}{c}\right)(\varepsilon_c)^r c^{r/2} \sum_{d(c)}^{*} \left(\frac{d}{c}\right)^r e\left(-n\frac{d}{c}\right)$$

$\boxed{\text{Suppose } r \text{ even}}$

$\Rightarrow$ if s is st. $(s, 2\det A) = 1$, let $D = (-1)^{r/2}\det A$, then

$$\sum_{c\mid s} \frac{g_c(n,Q)}{c^r} = \sum_{q\mid(n,s)} \left(\frac{D}{q}\right) q^{1-k} \prod_{p\mid \frac{s}{q}} \left(1 - \left(\frac{D}{p}\right)p^{-k}\right)$$

If $n \mid s$ and each prime exponent of s is strictly larger than that of n

$$\sum_{c\mid s} \frac{g_c(n,Q)}{c^r} = \prod_{p\mid s}\left(1 - \left(\frac{D}{p}\right)p^{-k}\right) \cancel{\prod_{p\mid n}\left(1 + \left(\frac{D}{p}\right)p^{-k}\right)} \sum_{q\mid n} \left(\frac{D}{q}\right) q^{1-k}$$

Eg take $s = p^{\operatorname{ord}_p n + 1}$ then $\longrightarrow$

$$\delta_p(n,Q) = \left(1 - \left(\tfrac{D}{p}\right) p^{-k}\right)\left(1 - \left(\tfrac{D}{p}\right) p^{1-k}\right)^{-1}\left(1 - \left(\tfrac{D}{p^{\operatorname{ord}_p n+1}}\right) p^{(\operatorname{ord}_p n+1)(1-k)}\right)$$

7/7

For any $p \nmid 2D$.

Hence $\delta_p(n,Q) \neq 0$ if $p \nmid 2D$.

$$\text{So } \delta(n,Q) = \prod_{p \mid 2D} \delta_p(n,Q)\, L\left(k, \left(\tfrac{4D}{\cdot}\right)\right)^{-1} \sum_{d \mid n} \left(\tfrac{4D}{d}\right) d^{1-k}.$$

where $L(s, (\frac{4D}{\cdot}))$ is the Dirichlet L-function $\sum_{n \geq 1} \frac{(\frac{4D}{n})}{n^s}$.

$$\delta_p(n,Q) = \sum_{i \geq 0} \frac{g_{p^i}(n,Q)}{p^{is}}$$

We have: FACT:

$$\delta_p(n,Q) \neq 0 \quad \forall\, p \mid 2\det A$$

$$\iff Q(m) \equiv n \quad (2^7 (\det A)^3).$$

Assuming this is solvable, we have that

$$\prod_{p \mid n}\left(1 + \left(\tfrac{D}{p}\right) p^{1-k}\right) \ll \delta(n,Q) \ll \prod_{p \mid n}\left(1 + \left(\tfrac{D}{p}\right) p^{1-k}\right)$$

These are constants if $r > 4$, if $r = 4$ they can vary $(\log\log 3n)^{-1} \ll \;\ll \log\log 3n)$.

If r is odd, We can also compute local densities but it is more complicated.

Theorem If $r \geq 4$ then any n sufficiently large for which the above congruence is solvable is represented by the Quadratic form Q. The number of representations is

$$n^{k-1} \ll r(n,Q) \ll n^{k-1} \quad \text{if } r \geq 5$$

$$n \prod_{p \mid n}\left(1 + \left(\tfrac{D}{p}\right)\tfrac{1}{p}\right) \ll r(n,Q) \ll n \prod_{p \mid n}\left(1 + \left(\tfrac{D}{p}\right)\tfrac{1}{p}\right) \text{ if } r = 4.$$