Classical Modular Forms Lecture 3 Sept 28. 2017.

Recall $GL_2^+(\mathbb{R}) \curvearrowright \mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ by

$$g \cdot z = \frac{az+b}{cz+d}.$$

Today we study the action of $SL_2\mathbb{Z}$ and certain finite-index subgroups $\Gamma \leq SL_2\mathbb{Z}$.

Definition: (Principal congruence subgroup)

For $q \geq 1$ an integer, let

$$\Gamma(q) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z} \;:\; \begin{array}{l} a \equiv d \equiv 1 \ (q) \\ c \equiv b \equiv 0 \ (q) \end{array} \right\}$$

Exercise: $\Gamma(q)$ is a normal subgroup of $SL_2\mathbb{Z}$ and its index is

$$[SL_2\mathbb{Z} : \Gamma(q)] = |SL_2(\mathbb{Z}/q\mathbb{Z})| = q^3 \prod_{p \mid q} \left(1 - \frac{1}{p^2}\right)$$

Definition: A subgroup $\Gamma \leq SL_2\mathbb{Z}$ is called a congruence or arithmetic subgroup if it contains some $\Gamma(q)$.

In particular, congruence subgroups are finite index in $SL_2\mathbb{Z}$.

Example: Hecke-Iwahori groups:

$$\Gamma_0(q) := \left\{ \gamma \in SL_2\mathbb{Z} : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod q \right\}$$

Example: Other important congruence subgroups include

$$\Gamma_1(q) := \left\{ \gamma \in SL_2\mathbb{Z} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod q \right\}$$

$$\Gamma_d(q) := \left\{ \gamma \in SL_2\mathbb{Z} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \bmod q \right\}.$$

Exercise: Compute the indices of $\Gamma_0(q), \Gamma_1(q), \Gamma_d(q)$ in $SL_2\mathbb{Z}$.

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ **Theorem** $SL_2\mathbb{Z}$ is generated by T and S.

Proof Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$

We proceed by induction on $|c|$.

Base case: $c = 0$.

Then $\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, but $S^2 = -\mathrm{Id}$, so

$$\gamma = T^b \text{ or } S^2 T^{-b}$$ ~~for some b ∈ Z~~.

Now assume that all $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ ~~are~~ with $|c'| < |c|$ are expressible as a word in S, T.

Note $T^k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+ck & b+dk \\ c & d \end{pmatrix}$, so by picking an appropriate k, ~~we~~ it suffices to show that the $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|a| < |c|$ are expressible in terms of S, T.

But $S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & b-d \\ a & b \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ ~~has~~

has $|c'| < |c|$ in this case, so is expressible in terms of S, T by induction hyp.

Then so is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, QED.

~~Congruence subgroups finitely generated~~ ?

Definition Let $\Gamma \subseteq SL_2\mathbb{R}$ be a discrete subgroup.

An open set $\mathcal{D} \subseteq \mathcal{H}$ is called a <u>fundamental domain</u> for Γ if

① $\forall z \in \mathcal{H}$, $\Gamma \cdot z \cap \overline{\mathcal{D}} \neq \emptyset$,

i.e. the closure of $\mathcal{D}$ meets every Γ-orbit $\Gamma . z$ in at least one point.

AND

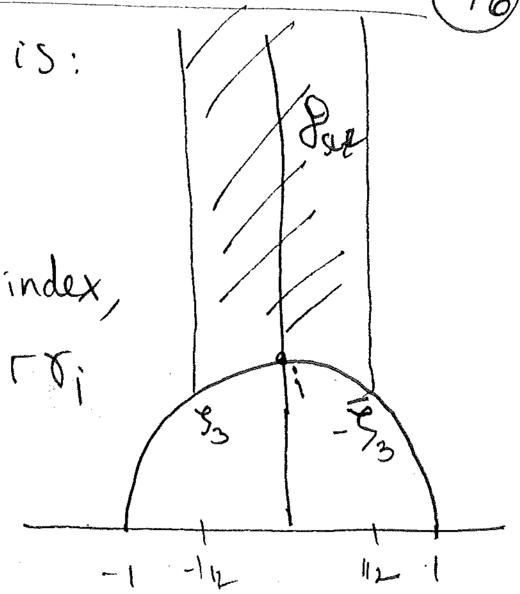
② $\forall z \in \mathcal{H}$, $|\Gamma . z \cap \mathcal{D}| \leq 1$ i.e. every orbit meets $\mathcal{D}$ in at <u>most</u> one point.

Exercise Equivalent conditions to be a fundamental domain are:

(1) $\mathcal{H} = \Gamma . \overline{\mathcal{D}}$ and (2) $\forall \gamma \in \Gamma$, $\gamma \neq \pm \mathrm{Id}$, $\mathcal{D} \cap \gamma . \mathcal{D} = \phi$

Theorem A fundamental domain for $SL_2\mathbb{Z}$ is:

$$\mathcal{D}_{SL_2\mathbb{Z}} = \{ z \in \mathcal{H} : |x| \leq 1/2 , |z| > 1 \}.$$



More generally For $\Gamma \subseteq SL_2\mathbb{Z}$ of finite index, let γ_i be coset representatives: $SL_2\mathbb{Z} = \bigcup_{\gamma_i} \Gamma \gamma_i$

then a fundamental domain is given by

$$\mathcal{D}_\Gamma = \bigcup_{\gamma_i} \gamma_i \mathcal{D}_{SL_2\mathbb{Z}}$$

PROOF Second statement: Follows directly from the exercise. (additional exercise).

Idea for first statement: Consider points in the orbit $SL_2\mathbb{Z} \cdot z$ with maximal imaginary part.

(1) want to show: $\forall z \in \mathcal{H}$, $SL_2\mathbb{Z} . z \cap \overline{\mathcal{D}} \neq \phi$

Let $z \in \mathcal{H}$. Claim There exists $\gamma \in SL_2\mathbb{Z}$ such that $\operatorname{Im} \gamma z$ is max'l.

Indeed, $\operatorname{Im} \gamma z = \dfrac{\operatorname{Im} z}{|cz+d|^2}$, so it suffices to find c, d such that $|cz+d|^2$ is minimal.

In fact, for each z, $|cz+d|^2 = (cx+d)^2 + (cy)^2 = c^2(x^2+y^2) + 2cdx + d^2 \geq 0$ is a positive-definite quadratic form in c, d.

The set $B_z(r) := \{ (c,d) \in \mathbb{R}^2 : |cz+d| \leq r \}$ is the convex hull of some ellipse centered at the origin. In particular it is compact, so $1 \leq |\mathbb{Z}^2 \cap B_z(r)| < \infty$ and is an integer.

$\lim_{r \to \infty} |\mathbb{Z}^2 \cap B_z(r)| = \infty$, $= 1$ for r sufficiently small, and is monotone increasing, so $\exists$ minimal r such that $|\mathbb{Z}^2 \cap B_z(r)| > 1$.

Thus there exists a $(c,d)_{\neq(0,0)}$ such that $|cz+d|$ is minimal.

Also observe that $(c,d)=1$:

Indeed, if $c=(c,d)\,c'$
$d=(c,d)\,d'$ ~~with~~ then $(c',d')=1$, and

$|cz+d| = (c,d)\,|c'z+d'|$, which contradicts minimality unless $(c,d)=1$.

Then by "Bézout's Identity" $\exists\ a,b\in\mathbb{Z}$ such that $ad-bc=1$.

So let $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2\mathbb{Z}$, and then γz has <u>maximal</u> imaginary part in the orbit $SL_2\mathbb{Z}.z$.

Let $z_0=\gamma z$, which has max'l imaginary part.

Then $\mathrm{Im}(Sz_0)=\dfrac{\mathrm{Im}\,z_0}{|z|^2}\le \mathrm{Im}\,z$, so $|z|\geq 1$.

and $T^k z_0 = z_0+k$, which doesn't change $\mathrm{Im}\,z_0$, so we can assume $\mathrm{Re}(z)\in(-1/2,1/2]$. Thus $z_0\in\mathcal{D}_{SL_2\mathbb{Z}}$.

(2) W.T.S. $\forall z\in\mathcal{H}$, $|SL_2\mathbb{Z}.z\cap\mathcal{D}|\le 1$.

Let $z\in\overline{\mathcal{D}}$. We will show for any $z'=\gamma z$, $z'\neq z$ s.t. $z'\in\overline{\mathcal{D}}$, $\gamma\in SL_2\mathbb{Z}$

then $z,\bar{z}\in\partial\mathcal{D}$.

We claim that $|cz+d|\geq 1$ for all $(c,d)=1$.

If $c=0$ or $d=0$, then the other is $\pm1 \implies$ Done.

So can assume $cd\neq 0$. Then:

$$|cz+d|^2 = (cx+d)^2+(cy)^2 = c^2(x^2+y^2)+2cd\,x+d^2$$
$$\underset{(\text{since } z\in\overline{\mathcal{D}})}{\geq} c^2-|cd|+d^2$$
$$\geq 2|cd|-|cd| = |cd|\geq 1 \quad \text{QED claim.}$$

Thus $\mathrm{Im}\,\gamma z = \mathrm{Im}\,z/|cz+d|^2 \le \mathrm{Im}\,z$, so $\mathrm{Im}\,z$ is max'l in $SL_2\mathbb{Z}.z$.

A modular function is a function $f: \mathcal{H} \to \mathbb{C}$ which satisfies this complicated semi-invariance property with respect to this action.

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We can clean up this definition by changing the action.

Let $\gamma \in \Gamma_0(4)$ and define $f|_k \gamma$ by $f|_k\gamma(z) = j_{1/2}(\gamma, z)^{-2k} f(\gamma z)$

$*$

Then $|_k \gamma$ defines a ~~left~~ right action on the space of functions $\{f: \mathcal{H} \to \mathbb{C}.\}$.

So modular functions are exactly those $f: \mathcal{H} \to \mathbb{C}$ for which $\exists q \geq 1$ st

$$f|_k \gamma = f \quad \forall \gamma \in \Gamma(4q).$$

For $k \in \mathbb{Z}_{\geq 0}$ we can simplify $*$:

$$f|_k\gamma(z) = \frac{(\det \gamma)^{k/2}}{(cz+d)^k} f(\gamma z) \qquad (**)$$

If $k \in \mathbb{Z}_{\geq 0}$ then $(**)$ defines ~~an~~ a right action of $GL_2^+ \mathbb{Q}$ on the space of functions with diagonal matricies acting trivially.

Definition If $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ then we take $*$ as the definition of $|_k \gamma$.

If $k \in \mathbb{Z}_{\geq 0}$ then we take $(**)$ as the definition of $|_k \gamma$.

$|_k \gamma$ is called the slash operator and defines a ~~left~~ right action on $\{f: \mathcal{H} \to \mathbb{C}\}$.

Definition Let $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. A modular function of weight k is a $f: \mathcal{H} \to \mathbb{C}$ for which there exists a $q \geq 1$ ($4 | q$ if $k \notin \mathbb{Z}$) such that for all $\gamma \in \Gamma(q)$

$$f|_k \gamma = f.$$

Let $\mathcal{F}_k(\Gamma(q))$ be the $\mathbb{C}$-VS of modular functions of weight k and level q.

Let $\mathcal{F}_k = \bigcup_{q \geq 1} \mathcal{F}_k(\Gamma(q))$ be the $\mathbb{C}$-VS of modular functions of weight k.

A modular form is a modular function which is holomorphic and which satisfies a certain growth condition...

Now let $\gamma z = z' \neq z$, $\gamma \in SL_2\mathbb{Z}$, and suppose $z, \bar{z} \in \bar{\mathcal{F}}$.

Then: $\operatorname{Im} z = \operatorname{Im} z' = \operatorname{Im} \gamma z$, and $|cz+d| = 1$.

- If $c = 0$ then $\gamma = \pm T^b$, $b \in \mathbb{Z}$, and $z' = z + b$.

  Then $|b| = 1$ and $\operatorname{Re}(z) = \pm 1/2$ and $\operatorname{Re}(z') = \mp 1/2$, so $z, z' \in \partial\mathcal{F}$.

- If $d = 0$ then $|c| = 1$, and $|z| = 1$, so $z, z' \in \partial\mathcal{F}$.
- If $cd \neq 0$ then $|cz+d| = |cd| = 1$ (previous calculation).

  So $|c| = |d| = 1$, so $x = \frac{-cd}{2} = \pm 1/2$, so $|z \pm 1| = 1$, so $y = \frac{\sqrt{3}}{2}$,

  Thus $z, z' = \zeta_3, -\bar{\zeta}_3 \in \partial\mathcal{F}$. Q.E.D.

Modular Functions

Recall $\left(\frac{c}{d}\right)$ the Jacobi symbol, and $\varepsilon_d = \begin{cases} 1 & d \equiv 1 \ (4) \\ i & d \equiv 3 \ (4) \end{cases}$

Version ①: Let $k \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ a half-integer.

A modular function of weight k is a function $f : \mathcal{H} \to \mathbb{C}$ for which there exists $q \geq 1$ such that for all $\gamma \in \Gamma(4q)$

$$f(\gamma z) = j_{1/2}(\gamma, z)^{2k} f(z)$$

where $j_{1/2}(\gamma, z) = \left(\frac{c}{d}\right) \varepsilon_d^{-1} (cz+d)^{1/2}$

We say f has weight k, level $4q$ in this case.

In particular, a modular function f is determined by its values on $\mathcal{F}_{\Gamma(4q)}$.

Example: (from last lecture) $\Theta(z) = \sum_{n \in \mathbb{Z}} e(n^2 z)$ is a weight $1/2$, level 4 modular function.

For Γ some group, there is an action on $\{f : \mathcal{H} \to \mathbb{C}\}$ by

$f \mapsto f \circ \gamma$, i.e. $f(z) \mapsto f(\gamma z)$.