The Θ-function $\Theta(z) = \sum_{n\in\mathbb{Z}} e(n^2 z)$ is a weight $1/2$, level 4 modular function, i.e. $\Theta|_{1/2}\gamma = \Theta \ \forall \gamma \in \Gamma(4)$.

It is not just a function $\mathcal{H} \to \mathbb{C}$ but a holomorphic function: The series converges uniformly on $\{z \in \mathcal{H} : \operatorname{Im} z \geqslant \delta\}$ for all $\delta > 0$.

It also satisfies a growth property:

Let $\gamma \in GL_2^+\mathbb{R}$, $z = x+iy \in \mathcal{H}$, then

$$\operatorname{Im} \gamma z = \det\gamma \cdot \frac{y}{|cz+d|^2}.$$

So, if f is a modular function of weight k, level q, then

$$z = x+iy \longmapsto F(z) := y^{k/2}|f(z)| \text{ is } \Gamma(q)\text{-invariant}.$$

Indeed, $|j_{1/2}(\gamma,z)| = |\left(\frac{c}{d}\right)\varepsilon_d^{-1}(cz+d)^{1/2}| = |cz+d|^{1/2}$, since $(c,d)=1$.

Then $F(\gamma z) = (\operatorname{Im}\gamma z)^{k/2}|f(\gamma z)| = \dfrac{y^{k/2}}{|cz+d|^k}|j_{1/2}(\gamma,z)|^{2k}|f(z)| = y^{k/2}|f(z)| = F(z)$

<u>Definition</u> A modular function f is of <u>polynomial</u> or <u>moderate</u> growth if there exists $A \geqslant 0$ such that for $z = x+iy \in \mathcal{H}$ and $F(z) = y^{k/2}|f(z)|$ there exists $C > 0$ such that

$$F(z) \leq C(y+y^{-1})^A \quad \forall z \in \mathcal{H}.$$

<u>Remark</u> (Vinogradov Notation) If X is a set, f a complex-valued function on X, g a positive real-valued function, $Y \subseteq X$ we say

$$f \ll g \quad \text{for } x \in Y$$

if there exists $C \geqslant 0$ such that $|f(x)| \leq C g(x) \ \forall x \in Y$.

So $F(z) \ll (y+y^{-1})^A \quad \forall z \in \mathcal{H}$.

<u>Remark</u>: We don't need a growth condition on $x = \operatorname{Re}(z)$. Indeed, $\begin{pmatrix}1&q\\0&1\end{pmatrix} \in \Gamma(q)$ so if f is modular of level q then $f(z) = f(z+q)$, so f is <u>periodic</u> in x of period q, so is determined by its restriction to any $x_0 \leq \operatorname{Re}(z) < x_0 + q$.

Definition A function $f: \mathcal{H} \to \mathbb{C}$ is called a holomorphic modular form if it

① Is a modular function for some weight k and level q
② Is holomorphic on $\mathcal{H}$
③ Is of moderate growth.

If the associated function $F(z)$ is bounded, then f is called a cusp form or cuspidal or parabolic or eine Spitzenform

$$M_k := \{\text{modular forms of weight } k\}$$
$$S_k := \{\text{cusp forms of weight } k\}$$
$$M_k(\Gamma) := \{f \in M_k : f|_k \gamma = f \quad \forall \gamma \in \Gamma\}$$
$$S_k(\Gamma) := \{f \in S_k : f|_k \gamma = f \quad \forall \gamma \in \Gamma\}.$$

$$\Gamma \leq \begin{cases} GL_2^+\mathbb{R} & k \in \mathbb{Z} \\ \Gamma_0(4) & k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}. \end{cases}$$

Example: The Θ-function is a modular form of weight $1/2$, level 4.
Need to check the growth condition:

$$\left| y^{1/4} \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z} \right| \leq y^{1/4} + 2y^{1/4} \sum_{n \geq 1} e^{-2\pi n^2 y} \leq y^{1/4} + 2y^{1/4} \int_0^\infty e^{-2\pi u^2 y}\, du$$
$$= y^{1/4} + \sqrt{\frac{2}{\pi}}\, y^{-1/4} \int_0^\infty e^{-v^2}\, dv$$
$$= y^{1/4} + \frac{1}{\sqrt{2}\, y^{1/4}} \leq y^{1/4} + y^{-1/4}$$
$$\leq 2^{3/4}(y + y^{-1})^{1/4}.$$

Why impose growth condition? We need it crucially for:

Theorem For any q, the space $M_k(\Gamma(q))$ is finite-dimensional.

PROOF To be proved later.

This is analogous to the following familiar result of Fourier series:

Theorem For any $\lambda \in \mathbb{R}$, the space $\mathcal{F}_\lambda(\mathbb{R}/\mathbb{Z})$ of smooth functions on $\mathbb{R}$ which are:

① $\mathbb{Z}$-periodic: $f(x+n) = f(x) \quad \forall n \in \mathbb{Z}$
② Eigenfunctions of the "Laplace" operator $\frac{\partial^2}{\partial x^2} f(x) = -\lambda f(x)$

is finite-dimensional.

In particular, $\mathcal{J}_\lambda(\mathbb{R}/\mathbb{Z}) = \{0\}$ if $\lambda \neq k^2$, $k \in \mathbb{Z}$, and

$$\mathcal{J}_{k^2}(\mathbb{R}/\mathbb{Z}) = \mathbb{C}\, e(kx) + \mathbb{C}\, e(-kx) \text{ if } k \in \mathbb{Z}.$$

Invariance by larger groups:

Recall $\Theta|_{1/2}\gamma = \Theta$ for all $\gamma \in \Gamma_0(4) \leq \Gamma(4)$.

Typically, we work with the larger groups $\Gamma_0(q), \Gamma_1(q), \Gamma_d(q)$.

In fact, the theory of modular forms for any of these groups reduces to that of $\Gamma_0(q)$, so we will discuss $\Gamma_0(q)$-modular forms. First, consider $\Gamma_d(q)$.

<u>Proposition</u> $\Gamma_d(q)$ normalizes $\Gamma(q)$, and in particular, $\Gamma_d(q)$ acts on $\mathcal{J}_k(\Gamma(q))$ by: $\gamma \in \Gamma_d(q)$, $f \in \mathcal{J}_k(\Gamma(q))$, then

$$f \longmapsto f|_k\gamma \in \mathcal{J}_k(\Gamma(q)).$$

<u>Proof</u> First: $\Gamma_d(q) \to SL_2\mathbb{Z}/q$ with kernel $\Gamma(q)$.

Second: It suffices to check $f|_k\gamma \in \mathcal{J}_k(\Gamma(q))$, $\gamma \in \Gamma_d(q)$.

Indeed, let $\gamma' \in \Gamma(q)$. Then:

$$(f|_\gamma)|_{\gamma'} = f|_{\gamma\gamma'} = (f|_{\underbrace{\gamma\gamma'\gamma^{-1}}_{\in\Gamma(q)}})|_\gamma = f|_\gamma. \quad \text{QED}.$$

Next task is to decompose $\mathcal{J}_k(\Gamma(q))$ with respect to this action.

<u>Proposition</u> The homomorphism $\Gamma_d(q) \longrightarrow (\mathbb{Z}/q)^\times$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto d \bmod q$ induces

$$\Gamma(q)\backslash\Gamma_d(q) \simeq (\mathbb{Z}/q)^\times$$

<u>Proof</u> By a previous exercise, $SL_2\mathbb{Z} \twoheadrightarrow SL_2\mathbb{Z}/q$ is <u>surjective</u>.

So $\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \in SL_2\mathbb{Z}/q$ lifts to an element $\begin{pmatrix} \bar{d} & b \\ c & d \end{pmatrix} \in \Gamma_d(q)$

So $\Gamma_0(q) \twoheadrightarrow (\mathbb{Z}/q)^\times$, with kernel $\Gamma(q)$. QED

Therefore $G := (\mathbb{Z}/q\mathbb{Z})^\times$ acts on $\mathcal{F}_k(\Gamma(q))$ by $f|_{\Gamma(q)\gamma} = f|_\gamma$

We can diagonalize this action. The eigenvalues are indexed by $\hat{G} = \mathrm{Hom}(G, \mathbb{C})$, the group of characters of G.

Background: Characters of a finite abelian group.

G a finite abelian group.

$\chi : G \to \mathbb{C}^\times$ a group homomorphism (i.e. a character)

$\hat{G}$ is a group under multiplication of functions

A very algebraic object, but nonetheless, arises from studying

$\mathcal{F}(G) = \{f : G \to \mathbb{C}\}$, the vector space of complex-valued functions on G.

Let $\delta_g \in \mathcal{F}(G)$ be the function given by

$$\delta_g(g') = \begin{cases} 1 & \text{if } g' = g \\ 0 & \text{if } g' \neq g \end{cases}$$ the indicator function of $g' = g$ or Kronecker δ function.

$\{\delta_g : g \in G\}$ is a basis for $\mathcal{F}(G)$, so $\mathcal{F}(G)$ is dimension $|G| < \infty$.

$\mathcal{F}(G)$ also has the natural L^2 hermitian inner product:

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} = \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

and $\{\delta_g\}$ is an orthogonal basis for $\langle , \rangle$.

This is great, but we haven't used that G is a group yet.

G acts on itself by left multiplication:

$m_g : G \to G \qquad m_g(g') = gg'$

$T_g : \mathcal{F}(G) \to \mathcal{F}(G) \qquad T_g f(g') = f(gg')$ is a linear map.

Actually, $g \mapsto T_g$ defines a group homomorphism $G \to GL(\mathcal{F}(G))$.

Indeed, we can check : $T_e = Id_{\mathcal{F}(e)}$ and $T_{gg'} = T_{g'} \circ T_g = T_{g'g} = T_g \circ T_{g'}$

Key Lemma For any $g \in G$, T_g is an isometry for the inner product:

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$$\langle T_g f_1, T_g f_2 \rangle = \langle f_1, f_2 \rangle$$

PROOF ~~Pf~~. Change variables $g' \to g'g$ in definition of $\langle , \rangle$. QED.

Since G is abelian, $\{T_g : g \in G\}$ is a commuting family of isometries we have:

Theorem (Spectral Theorem, ~~Un~~ Isometric Version)

Let V be a non-zero finite-dim'l complex vector space, with a positive-definite hermitian form. ~~Let T: V → V be an isometry~~ Let $\{T_i\}$ be a family of commuting isometries $T_i : V \to V$. ~~Then~~ $T_i T_j = T_j T_i \ \forall i,j$.

Then V has an orthogonal basis consisting of simultaneous eigenvectors of all T_i.

Applying this Theorem, we get eigenvectors $\{\psi\}$, say and

$$\mathcal{F}(G) = \bigoplus_{\psi} \mathbb{C}\psi \ , \text{ for which } T_g(\psi) = \chi_\psi(g)\psi ,$$

where $\chi_\psi(g)$ is an eigenvalue $\in \mathbb{C}$ for each g.

I claim $\chi_\psi(g) \in \hat{G}$. Indeed $\chi_\psi(e) = 1$ since $T_e = Id_{\mathcal{F}(a)}$

and $\chi_\psi(gg') = \chi_\psi(g)\chi_\psi(g')$ since $T_{gg'} = T_g \circ T_{g'}$.

Since T_g is an isometry, we have $|\chi_\psi(g)| = 1 \quad \forall g \in G$

Let $\mu_{|G|} := \{z \in \mathbb{C}^* : z^{|G|} = 1\}$.

By Lagrange's Thm $\chi_\psi(g)^{|G|} = \chi_\psi(g^{|G|}) = \chi_\psi(e) = 1$, so

$$\chi : G \to \mu_{|G|} .$$

Next ~~...~~ I claim that $\chi_\psi \in \mathbb{C}\psi$. (it is $\in \mathcal{F}(G)$ à priori).

$\forall g \in G$, $T_g \psi(e) = \psi(g) = \chi_\psi(g)\psi(e)$

$\psi \neq 0$, so $\exists g'$ st $\psi(g') \neq 0$, so $\psi(e) \neq 0$, so

$$\chi_\psi = \frac{1}{\psi(e)} \psi \in \mathbb{C}\psi.$$

Therefore, $\{\chi_\psi, \psi\}$ is an orthogonal family of $\mathcal{F}(G)$, and

$$\langle \chi_\psi, \chi_\psi \rangle = \frac{1}{|G|} \sum_{g \in G} |\chi_\psi(g)|^2 = 1, \text{ so}$$

$\{\chi_\psi, \psi\}$ is an orthonormal basis of eigenvectors of $\mathcal{F}(G)$ for $\{T_g : g \in G\}$.

normalized to have $\chi_\psi(e) = 1$.

So $\{\chi_\psi : \psi\} \subseteq \mathrm{Hom}(G, \mathbb{C}^\times) = \hat{G}$.

But also have the converse (idea is to use isometries):

let $\chi \in \hat{G}$, $\chi \neq 0$, so $\exists \psi$ st $\langle \chi, \psi \rangle \neq 0$

$$\langle \chi, \chi_\psi \rangle = \langle T_g \chi, T_g \chi_\psi \rangle = \chi(g)\overline{\chi_\psi(g)} \langle \chi, \chi_\psi \rangle$$

$$\Rightarrow \chi_\psi(g) = \chi(g)$$

Therefore $\hat{G} \subseteq \{\chi_\psi, \psi\}$, so $=$.

Therefore $|G| = |\hat{G}|$.

Double dual: $G \to \hat{\hat{G}}$ by $g \mapsto \eta_g$, where $\eta_g : \hat{G} \to \mathbb{C}^\times$

$$\eta_g(\chi) = \chi(g).$$

$g \mapsto \eta_g$ is injective thus bijective, so $\hat{\hat{G}} \simeq G$ canonically.

Exercise: Show $G \approx \hat{G}$ (non-canonically).

Hint: Show first for any finite abelian groups G_1, G_2 that

$$\widehat{G_1 \times G_2} \cong \widehat{G_1} \times \widehat{G_2}.$$

This reduces the problem to the case that G is cyclic.