Classical Modular Forms Lecture 5 Oct 5. 2017

Last time: G a finite abelian group

The characters $\hat{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ form a basis for $\mathcal{F}(G) := \{f : G \to \mathbb{C}\}$.

which is orthonormal, and is composed of joint eigenfunctions for the operators

$$\{T_g : g \in G\} \qquad T_g f(g') = f(gg').$$

Moreover, $G \simeq \hat{\hat{G}}$ and $G \approx \hat{G}$, the latter being non-canonical: it depends on arbitrary choices of generators for cyclic groups.

We have
$$\frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi'(g)} = \delta_{\chi = \chi'}$$

$$\frac{1}{|\hat{G}|} \sum_{\chi \in \hat{G}} \chi(g) \overline{\chi(g')} = \delta_{g = g'}$$

<u>Theorem</u> Let G an abelian group acting linearly on a complex vector space V. (i.e. $\exists\ \rho : G \to GL(V)$ a group homomorphism)

We have a direct sum decomposition

$$V = \bigoplus_{\chi \in \hat{G}} V_\chi,$$

where $V_\chi = \{x \in V : \rho(g)(x) = \chi(g) x\}$.

<u>PROOF</u> The spaces V_χ are clearly linearly disjoint for $\chi \neq \chi'$, since they have distinct eigenvalues.

$\bigoplus_{\chi \in \hat{G}} V_\chi \subseteq V$, but need to show the reverse inclusion.

For any $\chi \in \hat{G}$ there is a homomorphism

$$p_\chi : V \to V_\chi$$

$$p_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g)^{-1} \rho(g)$$

Indeed, check that the image of ρ_χ lands in V_χ:

$$\rho(g')\rho_\chi(x) = \rho(g') \frac{1}{|G|}\sum_{g\in G}\chi(g)\rho(g)(x) = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\rho(g'g)(x)$$

let $g = g'^{-1}g''$, $\longrightarrow$

$$= \frac{1}{|G|}\sum_{g''\in G}\chi(g'^{-1}g'')^{-1}\rho(g'')(x)$$

$$= \chi(g')\rho_\chi(x).$$

By the first orthogonality relation:

$v \in V_{\chi'}$, $\chi' \in \hat{G}$

$$\rho_\chi v = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\rho(g)v = \frac{1}{|G|}\sum_{g\in G}\chi(g)^{-1}\chi'(g) = \delta_{\chi=\chi'}$$

$$\text{so } \rho_\chi\big|_{V_\chi} = \begin{cases} Id_{V_\chi} & \chi=\chi' \\ 0 & \chi\neq\chi' \end{cases}$$

and by the second

$$\sum_{\chi\in\hat{G}}\rho_\chi = Id_V, \quad \text{indeed} \quad \sum_{\chi\in\hat{G}}\rho_\chi = \sum_{g\in G}\frac{1}{|G|}\sum_{\chi\in\hat{G}}\chi(g)^{-1}\rho(g)$$

$$= \sum_{g\in G}\delta_{g=e}\rho(g) = \rho(e) = Id_V.$$

QED.

Dirichlet Characters

Suppose $G = (\mathbb{Z}/q)^\times$ for some $q \geq 1$.

Given $\chi \in \widehat{(\mathbb{Z}/q)^\times}$, i.e. $\chi: (\mathbb{Z}/q)^\times \longrightarrow \mathbb{C}^\times$, one extends to a function on $\mathbb{Z}/q$ which we also (abusively) call χ, by setting $\chi(n) = 0$ if $(n,q) > 1$

These are called Dirichlet characters (perhaps you saw them in the last exercise class)

Eg.

n	0	1	2
χ_0	0	1	1
χ	0	1	-1

or

n	0	1	2	3	4	5
χ_0	0	1	0	0	0	1
χ_1	0	1	0	0	0	-1

So, the extended Jacobi symbol $\left(\frac{\cdot}{q}\right)$ is ~~the~~ a real-valued Dirichlet character mod q. (q odd, $q \geq 1$)

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The Jacobi symbol $\left(\frac{d}{\cdot}\right)$ is a ~~...~~ real-valued Dirichlet character mod $4|d|$.

We apply the Theorem to $\mathcal{F}_k(\Gamma(q))$. $(\mathbb{Z}/q)^\times \simeq \Gamma_d(q)/\Gamma(q)$ acts via $|_k \gamma$.

So $\mathcal{F}_k(\Gamma(q)) = \bigoplus_{\chi \bmod q} \mathcal{F}_k(\Gamma(q))_\chi$ Dirichlet characters mod q.

where $\mathcal{F}_k(\Gamma(q))_\chi = \{ f \in \mathcal{F}(\Gamma(q)) : \forall \gamma \in \Gamma_d(q) \ \ f|_{k\gamma} = \chi(\gamma) f \}$

where $\chi : \Gamma_d(q) \to \mathbb{C}^\times$ is defined by $\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi(d)$
(check it is actually a character of $\Gamma_d(q)/\Gamma(q)$).
[χ is a Dirichlet character mod q]

Call $f \in \mathcal{F}_k(\Gamma(q))_\chi$ a modular function of <u>nebentypus χ</u>.

<u>Similarly</u>, modular forms or cusp forms.

<u>Extension to Hecke-Iwahori subgroup:</u>

Let $W_q = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$, we can calculate: $W_q \begin{pmatrix} a & b \\ c & d \end{pmatrix} W_q^{-1} = \begin{pmatrix} a & qb \\ c/q & d \end{pmatrix}$

Thus $W_q \Gamma_0(q^2) W_q^{-1} \leq \Gamma_d(q)$.

Given $f \in \mathcal{F}(\Gamma(q))_\chi$, let $f_q : \mathcal{H} \to \mathbb{C}$ be defined $f_q(z) := f(qz) = f(W_q z)$.

Claim: f_q is a $\Gamma_0(q^2)$-modular function. Let $\gamma = \begin{pmatrix} a & b \\ cq^2 & d \end{pmatrix} \in \Gamma_0(q^2)$

$\boxed{k \in \mathbb{Z}}$

$$f_q|_k \gamma = (cq^2 z + d)^{-k} f(W_q \gamma z) = (cq(qz) + d)^{-k} f(W_q \gamma W_q^{-1} qz)$$
$$= f|_{W_q \gamma W_q^{-1}} (qz) = \chi(d) f(qz) = f_q(z)\chi(d)$$

So f_q is $\Gamma_0(q^2)$ modular of nebentype character χ.

likewise if $k \in \frac{1}{2}\mathbb{Z} - \mathbb{Z}$ ($4 \mid q$ in this case).

$$f_q|_{k}\gamma(z) = \left(\frac{cq^2}{d}\right)^{-2k} \varepsilon_d^{-2k} (cq^2 z + d)^{-k} f(q\gamma z) = \left(\frac{q}{d}\right)^{2k} \left(\frac{cq}{d}\right)^{-2k} \varepsilon_d^{-2k} (cq(qz)+d)^{-k} f(W_q \gamma W_q^{-1} qz)$$

$$= \left(\frac{q}{d}\right) f|_{k W_q \gamma W_q^{-1}} (qz) = \left(\frac{q}{d}\right) \chi(d) f_q(z),$$

so f is modular for $\Gamma_0(q)$ and nebentypus $\left(\frac{q}{d}\right)\chi(d)$.

Consider the space

$$\mathcal{F}_k(q,\chi) := \mathcal{F}_k(\Gamma_0(q))_\chi = \{ f : \mathcal{H} \to \mathbb{C} \text{ s.t. } \forall \gamma \in \Gamma_0(q) \quad f|_k\gamma = \chi(\gamma) f \}.$$

So the study of $\mathcal{F}_k(\Gamma(q))$ is reduced to studying $\mathcal{F}(q^2, \chi)$ or $\mathcal{F}(q^2, (\frac{q}{\cdot})\chi)$ forall $\chi \bmod q$.

Therefore it is typical to only consider $\mathcal{F}_k(q,\chi)$, $q \geq 1$, $\chi \bmod q$.

Note: $-\mathrm{Id} \in \Gamma_0(q)$, so if $\chi(-1) \neq (-1)^k$, then $\mathcal{F}_k(q,\chi) = 0$

Therefore it is typical to assume that $\chi(-1) = (-1)^k$

Examples: $\Theta \in \mathcal{M}_{1/2}(\Gamma_0(4))$ has trivial central character.

Define $\Theta_\ell(z) : \Theta(z)^\ell$ $\quad \ell \geq 1$ an integer.

$$\forall \gamma \in \Gamma_0(4) \quad \Theta_\ell(\gamma z) = \left(\left(\frac{c}{d}\right)\varepsilon_d^{-1}\right)^\ell (cz+d)^{\ell/2} \Theta_\ell(z)$$

$$= \chi_4(\gamma)^{\ell/2} (cz+d)^{\ell/2} \Theta_\ell(z)$$

$$\text{where } \chi_4(\gamma) = \left(\frac{c}{d}\right)^2 \varepsilon_d^{-2} = \underbrace{\chi_4(d)}_{\text{a Dirichlet character mod } 4} = \begin{cases} 1 & d \equi 1 \ (4) \\ -1 & d \equiv 3 \ (4) \end{cases}$$

So $\Theta_\ell(z)$ is a holomorphic modular form of weight $\ell/2$ for $\Gamma_0(4)$ of nebentype character χ_4 $\in \mathcal{M}_{\ell/2}(4,\chi_4)$
if $\ell \equiv 2 (4)$

if $\ell \equiv 0 (4)$, $\Theta_\ell(z)$ is a holomorphic modular form of weight $\ell/2$ for $\Gamma_0(4)$ and trivial character.
$\in \mathcal{M}_{\ell/2}(4, \text{triv}) = \mathcal{M}_{\ell/2}(\Gamma_0(4))$.

More basic examples, level 1 modular forms. $4 \nmid 1$, so $k \in \mathbb{Z}$, and χ must be mod 1, so trivial, and $\chi(-1) = (-1)^k \Rightarrow M_k(SL_2\mathbb{Z}) = 0$ if k is odd.

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Suppose $k > 2$. We have the Eisenstein series:

$$E_k(z) := \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k}.$$

$\boxed{\text{Exercise!}}$ Since $k > 2$, $E_k(z)$ converges absolutely, and uniformly on compact subsets $\Omega \subseteq \mathcal{H}$.

Therefore $E_k(z)$ defines a holomorphic function on $\mathcal{H}$.

Moreover, we may rearrange at will.

Note $E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2\mathbb{Z}} \frac{1}{j(\gamma,z)^k}$, and $\frac{j(\gamma\gamma', z)}{j(\gamma', z)} = j(\gamma, \gamma' z)$

So $E_k(\gamma' z) = \sum_{\gamma \in \Gamma_\infty \backslash SL_2\mathbb{Z}} \frac{j(\gamma',z)^k}{j(\gamma\gamma',z)^k} = j(\gamma',z)^k \frac{1}{2} \sum_{\gamma''} \frac{1}{j(\gamma'',z)^k} = j(\gamma',z)^k \cdot E_k(z)$.

$\gamma'' = \gamma\gamma'$

$\Rightarrow E_k$ is modular of weight k.

Note that if k is odd, then $E_k = 0$, since the terms cancel symmetrically.

To show E_k is of moderate growth, we compute its Fourier expansion.

Define also

$$G_k(z) := \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ (c,d) \neq 0}} \frac{1}{(cz+d)^k} = \sum_{g \geq 1} \sum_{\gcd(c,d)=g} \frac{1}{(cz+d)^k}$$

$$= \sum_{g \geq 1} \frac{1}{g^k} \sum_{\gcd(c',d')=1} \frac{1}{(c'z+d')^k}$$

$$= 2\zeta(k) E_k(z).$$

So it suffices to study $G_k(z)$.

Exercise Use $\sin \pi z = \pi z \prod_{n=1}^{\infty} (1 - z^2/n^2)$ to show that

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left[\frac{1}{z-d} + \frac{1}{z+d}\right] = \pi \cot \pi z = \pi i - 2\pi i \sum_{m=0}^{\infty} e(mz).$$

Differentiate $k-1$ times with respect to z $\Rightarrow$

$$\sum_{d\in\mathbb{Z}} \frac{1}{(z+d)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1} e(mz) \quad , k\geq 2$$

If $k>2$ even we have

$$G_k(z) = \sum_{(c,d)\neq(0,0)} \frac{1}{(cz+d)^k} = \sum_{d\neq 0}\frac{1}{d^k} + 2\sum_{c=1}^{\infty}\left(\sum_{d\in\mathbb{Z}}\frac{1}{(cz+d)^k}\right)$$

$$= 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{c=1}^{\infty}\sum_{m=1}^{\infty} m^{k-1} e(cmz)$$

$$\Rightarrow G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e(nz) \qquad k>2 \text{ even},$$

where $\sigma_{k-1}(n) = \sum_{\substack{m\mid n\\ m>0}} m^{k-1}$. (the Fourier coefficients are Arithmetic in nature)

Note then $E_k(z) = 1 + \frac{(2\pi i)^k}{\zeta(k)(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\,e(nz)$ $\qquad k>2$ k even.

Fact (complex analysis?) $\zeta(k)\in\pi^k\mathbb{Q}$, k even ≥ 2 (Bernoulli numbers.... p-adic L-fcns...)

So the Fourier coeffs of $E_k(z)$ are RATIONAL numbers.

$\lim_{\operatorname{Im}(z)\to\infty} E_k(z) = 1$, so $y^{k/2}|E_k(z)|$ is of moderate growth

Thus: $E_k(z)$ is a holomorphic modular form of weight k for $SL_2\mathbb{Z}$

To be shown later: $\dim M_k(SL_2\mathbb{Z}) = \begin{cases} 1 & \text{if } k=0\\ 0 & \text{if } k=2\\ 1 & \text{if } 4\leq k\leq 10 \text{ even}\\ 0 & \text{if } k \text{ odd}\end{cases}$

These dimension formulas have arithmetic consequences, for example $E_4^2 = E_8$ $\Rightarrow$

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{i=1}^{n-1}\sigma_3(i)\,\sigma_3(n-i) \quad \text{for all } n\geq 1.$$

e.g. $\frac{E_4^3 - E_6^2}{1728} =: \Delta$ "Ramanujan Δ-fcn" $\Delta\in S_{12}(SL_2\mathbb{Z})$, so $M_{12}(SL_2\mathbb{Z})\geq 2$

Back to θ-functions :

$$\Theta_\ell(z) = \sum_{(n_1,\dots,n_\ell)\in\mathbb{Z}^\ell} e((n_1^2+n_2^2+\dots n_\ell^2)z) = \sum_{n\geq 0} r_\ell(n)\, e(nz)$$

where $r_\ell(n) = |\{n_1^2+\dots+n_\ell^2 = n\}|$

$r_\ell(n) = \#$ of representations of n as a sum of l squares of integers.

Alternatively, let $R_\ell(n) = \{x=(x_1,\dots,x_\ell)\in\mathbb{Z}^\ell : Q_\ell(x)=n\}$

where $Q_\ell(x_1,\dots,x_\ell) = x_1^2+\dots+x_\ell^2$ is the Euclidean Quadratic form.

So $r_\ell(n) = \#$ of vectors in $\mathbb{R}^\ell$ with integer coordinates which lie on the sphere of radius $\sqrt{n}$.

~~Obsern~~ Using Eisenstein Series, Fourier expansions, dimension of $M_2(\Gamma_0(4))$ we can show

<u>Thm</u> (Lagrange 4 squares Theorem)

$$r_4(n) = 8\sum_{\substack{d\mid n\\ 4\nmid d}} d \quad n\geq 1$$

, in particular every positive number is expressible as a sum of 4 squares.

More generally, there is an Eisenstein series "at the cusp at ∞" for each $\Gamma_0(q)$.

let $\Gamma_\infty = \{\pm T^b : b\in\mathbb{Z}\}$ "stabilizer of cusp at ∞"

$k>2$

$$E_k(z) = \sum_{\gamma\in\Gamma_0(q)/\Gamma_\infty} \bar{\chi}(\gamma)\, j(\gamma,z)^{-k} \in M_k(\Gamma_0(q),\chi)$$

More generally, for each $m\geq 0$

$$P_m(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(q)} \bar{\chi}(\gamma)\, j(\gamma,z)^{-k} e(m\gamma z) \in M_k(\Gamma_0(q),\chi)$$

is called the m-th <u>Poincaré</u> series. Facts:

① $P_m\in S_k(\Gamma_0(q),\chi)$ if $m\geq 1$

$\{P_m : m\geq 0\}$ span $M_k(\Gamma_0(q),\chi)$.

We must do more systematically!
- Geometry of modular curves
- Dimension formulas
- Fourier expansions