Classical Modular Forms Lecture 6 12.10.2017

Let us fix one omission from last week's Lecture.

I claimed: $E_k(z) := \frac{1}{2}\sum_{(c,d)=1} \frac{1}{(cz+d)^k} = \sum_{\gamma \in \Gamma_\infty \backslash SL_2\mathbb{Z}} \frac{1}{j(\gamma,z)^k}$

for $k \geq 4$ even

where $\Gamma_\infty = \{\pm T^n : n \in \mathbb{Z}\}$ is the $\underbrace{\text{"stabilizer of the cusp at } \infty\text{"}}_{\text{We will explain this in more detail in following lectures.}}$

It suffices to check

$$\langle T\rangle \backslash SL_2\mathbb{Z} \xrightarrow[\text{bij}]{\sim} \{(c,d) \in \mathbb{Z}^2 : \gcd(c,d)=1\}.$$

Since $\pm Id$ acts trivially on $(cz+d)^k$ (k is even), it suffices to check the claim to prove the formula for the E_k Eisenstein series.

There is an evident map

$$SL_2\mathbb{Z} \xrightarrow{\varphi} \{(c,d)=1\}$$

since $ad-bc=1 \Rightarrow (c,d)=1$.

We claim the image only depends on the orbit $\langle T\rangle\gamma \subseteq SL_2\mathbb{Z}$.

Indeed, let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\left\{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} : n \in \mathbb{Z}\right\} = \left\{\begin{pmatrix} a+cn & b+nd \\ c & d \end{pmatrix} : n \in \mathbb{Z}\right\}$$

Thus φ is constant along left $\langle T\rangle$-orbits.

So $\langle T\rangle \backslash SL_2\mathbb{Z} \longrightarrow \{(c,d)=1\}$ is a function.

It is surjective: Given $(c,d)=1$ $\exists a,d \in \mathbb{Z}$ st $ad-bc=1$
"Bezout's formula"

and thus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$

it is injective: Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$ having same image.

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then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & b\alpha - a\beta \\ 0 & 1 \end{pmatrix} \in \langle T \rangle$.

So $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ c & d \end{pmatrix}$ are in the same $\langle T \rangle$-orbit.

Thus $\langle T \rangle \backslash SL_2\mathbb{Z} \xrightarrow[\text{bij}]{\sim} \{(c,d) \in \mathbb{Z}^2 : \gcd(c,d) = 1\}$.

~~Rmk~~ More generally, we can define an Eisenstein series "at the cusp at ∞" for each $\Gamma_0(q)$, $k > 2$, $k \in \mathbb{Z}$

~~Let~~ $E_k(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(q)} \bar{\chi}(\gamma) j(\gamma, z)^{-k} \in M_k(\Gamma_0(q), \chi)$

by exactly the same proof.

~~Better~~: <u>Theta Functions</u>

Recall $\Theta_\ell(z) := \Theta(z)^\ell$.

We have
$$\Theta_\ell \in \begin{cases} M_{\ell/2}(4, \chi_4) & \text{if } \ell \equiv 2(4) \\ M_{\ell/2}(4, \chi_0) & \text{if } \ell \equiv 0(4) \end{cases}$$

where

	0	1	2	3
χ_0	0	1	0	1
χ_4	0	1	0	-1

are the two Dirichlet characters module 4.

We have $\Theta_\ell(z) = \sum_{(n_1, \dots, n_\ell) \in \mathbb{Z}^\ell} e((n_1^2 + \dots + n_\ell^2) z) = \sum_{n \geq 0} r_\ell(n) e(nz)$

where $r_\ell(n) = |\{n_1^2 + \dots + n_\ell^2 = n\}|$

$= \#$ of representations of n as a sum of ℓ squares of integers.

Alternately: Let $R_\ell(n) = \{x = (x_1, \dots, x_\ell) \in \mathbb{Z}^\ell : Q_\ell(x) = n\}$

where $Q_\ell(x_1, \dots, x_\ell) = x_1^2 + \dots + x_\ell^2$ is the Euclidean Quadratic form.

So, $r_\ell(n) = \#$ of vectors in $\mathbb{R}^\ell$ with integer coordinates which lie on the sphere of radius $\sqrt{n}$. 3/8

Using Eisenstein series, Θ_4, ~~and~~ Fourier expansions, and the fact that $\dim M_2(\Gamma_0(4)) = 2$

we can show: ~~[illegible]~~

Theorem Lagrange 4 squares Theorem.

Every positive integer is expressible as a sum of four squares.

In fact:
$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d. \qquad n \geq 1.$$

Recall from the exercise session the weight two ~~ein~~ Eisenstein Series

$$G_2(z) := \sum_{c \in \mathbb{Z}} \left(\sum_{d \in \mathbb{Z}_c'} \frac{1}{(cz+d)^2} \right) \text{ where } \mathbb{Z}_c' = \begin{cases} \mathbb{Z} & c \neq 0 \\ \mathbb{Z} \setminus \{0\} & c = 0 \end{cases}$$

$$\text{or } \sum_{d \neq 0} \frac{1}{d^2} + \sum_{c \neq 0} \left(\sum_{d \in \mathbb{Z}} \frac{1}{(cz+d)^2} \right).$$

This does not converge absolutely. However, recall you also showed by Complex analysis

$$\sum_{d \in \mathbb{Z}} \frac{1}{(z+d)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m \geq 1}^{\infty} m^{k-1} e(mz)$$

is valid $\forall k \geq 2$.

Then
$$G_2 = 2\zeta(2) - 8\pi^2 \sum_{n=1}^{\infty} \sigma_1(n) e(nz) \qquad (*)$$

where we have not done any re-arrangement, nor used Fubini's Theorem.

In fact $(*)$ is now an absolutely and uniformly on compacta convergent formula for $G_2(z)$.

In fact: $G_2(z)$ is • Holomorphic on $\mathcal{H}$
• ~~Satisfies a growth~~
Is moderately growth
(thanks to this "Fourier Expansion").

We might as well take the Fourier expansion as the definition of G_2.

Also in exercises you showed: $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbb{Z}$

$$G_2 \Big|_2 \gamma (z) = G_2(z) - \frac{2\pi i c}{(cz+d)}, \text{ so } G_2(z) \text{ is } \underline{\text{NOT}}$$

a modular form.

Note this is precisely because we don't have absolute convergence of the series defining G_2.

~~However,~~ we Recall: $\dim \mathcal{M}_k(SL_2\mathbb{Z}) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 (12) \\ \lfloor k/12 \rfloor & k \equiv 2(12) \end{cases}$

So this makes sense. OTOH, let $N \geq 1$, let

$$G_{2,N}(z) := G_2(z) - N G_2(Nz).$$

<u>Claim</u>: $G_{2,N}(z) \in \mathcal{M}_2(\Gamma_0(N))$.

Clear that $G_{2,N}(z)$ satisfies the growth condition and is holomorphic so it suffices to check modularity.

$\forall \gamma \in \Gamma_0(N)$

$$G_{2,N} \Big|_2 \gamma (z) = G_2 \Big|_2 \gamma (z) - N G_2^{(N)} \Big|_2 \gamma (z), \quad \text{where } G_2^{(N)}(z) := G_2(Nz)$$

$$= G_2(z) - \frac{2\pi i c}{cz+d} - N \frac{G_2(N\gamma z)}{(cz+d)^2}$$

Recall $N\gamma N^{-1} = \begin{pmatrix} a & bN \\ c/N & d \end{pmatrix} \in SL_2\mathbb{Z}$, so

$$= G_2(z) - \frac{2\pi i c}{cz+d} - N\frac{G_2(N\gamma N^{-1}Nz)}{(\frac{c}{N}(Nz)+d)^2}$$

$$= G_2(z) - \frac{2\pi i c}{cz+d} - N\left(G_2(Nz) - \frac{2\pi i\, c/N}{(\frac{c}{N}(Nz)+d)^2}\right)$$

$$= G_2(z) - NG_2(Nz) = G_{2,N}(z)$$

QED.

So we have another nice collection of examples of modular forms of weight $\underline{2}$ for the congruence subgroup $\Gamma_0(N)$.

Note: $G_{2,1} = 0$, which makes sense given the dimension formula.

Proof of 4 squares theorem:

Use Fourier expansions:

$$G_{2,2}(z) = G_2(z) - 2G_2(2z)$$

$$= -2\zeta(2) - 8\pi\sum_{n\geq 1}\left[\sigma_1(n) - 2\delta_{2|n}\,\sigma(n/2)\right]e(nz).$$

$\sigma_1(n)$ is multiplicative: if $(n,m)=1$ then $\sigma_1(n)\sigma_1(m) = \sigma_1(nm)$.

So if n odd $\quad \sigma_1(n) - 2\delta_{2|n}\sigma(n/2) = \sigma_1(n)$

if n even, $\quad$ write $n = 2^\nu m$, where m is odd

$$\sigma_1(n) - 2\delta_{2|n}\sigma(n/2) = \sigma_1(m)\left[\sigma_1(2^\nu) - 2\sigma(2^{\nu-1})\right]$$

telescopes.

$$= \sigma_1(m)$$

so, $$G_{2,2}(z) = -2\zeta(2) - 8\pi^2\sum_{n\geq 1}\Bigg(\sum_{\substack{d|n\\ d\ \text{odd}}} d\Bigg)e(nz)$$

Similarly $$G_{2,4}(z) = -\cancel{2}\zeta(2) - 8\pi^2\sum_{n\geq 1}\Bigg(\sum_{\substack{d|n\\ 4\nmid d}} d\Bigg)e(nz)$$

Now note that $\quad G_{2,2} \in \mathcal{M}_2(\Gamma_0(2)) \subseteq \mathcal{M}_2(\Gamma_0(4))$

$G_{2,4} \in \mathcal{M}_2(\Gamma_0(4))$.

~~Note~~ Note : $\zeta(2) = \pi^2/6$,

~~$G_{2,2}(z)$~~ Now, we have $\dim M_2(\Gamma_0(4)) = 2$

(we will prove this later)

$G_{2,2}$ and $G_{2,4}$ are obviously linearly dependent, therefore form a Basis for $M_2(\Gamma_0(4))$.

But $\Theta_4 \in M_2(\Gamma_0(4))$, so $\Theta_4 = aG_{2,2} + bG_{2,4}$ for some $a, b \in \mathbb{C}$.

We have, since $\zeta(2) = \pi^2/6$

$$\Theta_4(z) = \sum_{n \geq 0} r_4(n) e(nz)$$
$$= 1 + 8e(z) + 24e(2z) + \cdots$$
$$-\frac{3}{\pi^2} G_{2,2}(z) = 1 + 24e(z) + \cdots$$
$$-\frac{1}{\pi^2} G_{2,4}(z) = 1 + 8e(z) + \cdots$$

Therefore $\Theta_4(z) = -\frac{1}{\pi^2} G_{2,4}(z)$.

By equating Fourier coeffs we have proved the 4 squares theorem

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Remarks: The 2, 4, 6, 8 squares problems also admit similar elegant exact formulas.

For all $l \geq 2$, there is an asymptotic solution: $\exists$ a "nice" function $\tilde{r}_l(n)$ such that $\lim_{n \to \infty} \frac{r_l(n)}{\tilde{r}_l(n)} = 1$.

~~Alge~~ Also generalizes to arbitrary Quadratic forms! (TBA...)
This will be addressed later in the course.

To prove dimension formulas, we need to develop more theory:
Fourier Expansions, and geometry of $\Gamma \backslash \mathcal{H}$ for Γ congruence.

Fourier Expansions

Let f be a modular form relative to Γ. Let Γ a congruence subgroup.

Then $\exists\, h > 0$ st $T^h = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$,

indeed, since $\Gamma(q) \subseteq \Gamma$ for some $q \geq 1$, we can take $h = q$.

We suppose h is minimal for this property:

Def The minimal such h is called "The width of the cusp of Γ at ∞".

We have $T^h f(z) = f(z+h) = f(z)$, i.e. for any $y > 0$, the function $x \mapsto f(x+iy)$ is h-periodic.

We have a Fourier expansion:

$$f(z) = \sum_{n \in \mathbb{Z}} c_f(n;y)\, e\left(n \frac{x}{h}\right).$$

with $c_f(n;y) = \frac{1}{h} \int_0^h f(iy+u)\, e\left(-n\frac{u}{h}\right) du.$

Since f is holomorphic it satisfies the Cauchy-Riemann eqns:

$$\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x},$$

So:
$$\frac{\partial}{\partial y} c_f(n;y) = \frac{1}{h} \int_0^h \frac{\partial}{\partial y} f(u+iy)\, e\left(-n\frac{u}{h}\right) du$$

$$= -\frac{i}{h} \int_0^h f(u+iy) \frac{\partial}{\partial u} e\left(\frac{-nu}{h}\right) du = -\frac{2\pi n}{h} c_f(n;y)$$

(integration by parts! periodicity!)

Thus $c_f(n;y) = a_f(n) e^{-2\pi n y/h} = a_f(n) e(niy/h)$

Therefore $$f(z) = \sum_{n\in\mathbb{Z}} a_f(n)\, e\left(n\frac{z}{h}\right)$$

with $$a_f(n) = e^{\frac{2\pi n y}{h}} \frac{1}{h}\int_0^h f(u+iy)\, e\left(-\frac{nu}{h}\right) du.$$

The function $y^{k/2}|f(z)| =: F(z)$ is of <u>moderate growth</u>, so for $n<0$ we have

$$|a_f(n)| \le c\, y^{-k/2}(y+y^{-1})^A e^{2\pi n y} \qquad \text{as } y\to\infty$$

Hence if $n<0$ then $|a_f(n)| = 0$.

<u>Definition</u> The expansion

$$f(z) = \sum_{n\geq 0} a_f(n)\, e\left(n\frac{z}{h}\right) \quad \text{is called}$$

the <u>Fourier expansion</u> of f at ∞ for Γ. The numbers $a_f(n)$ $n\geq 0$ are called the <u>Fourier coefficients</u> of f. The 0-th coefficient is called the constant term of this expansion.

We can give a bound for the Fourier coefficients:

<u>Prop</u> If A is the exponent appearing in the growth condition for f, then one has

$$|a_f(n)| \le c\, n^{k/2+A}.$$

In particular, if f is cuspidal,

$$|a_f(n)| \le c\, n^{k/2}.$$

<u>Proof</u> For $y>0$

$$|a_f(n)| \le c\, y^{-k/2}(y+y^{-1})^A e^{\frac{2\pi n y}{h}}.$$

Let $y = 1/n$. QED.

Therefore the Fourier expansion converges rapidly and $f(z) = a_f(0) + O\left(e^{-\frac{2\pi y}{h}}\right)$ as $y\to\infty$.