Classical Modular Forms Lecture 7 17.10.2017

Let $f \in M_k(\Gamma)$, let $h \in \mathbb{N}$ be minimal such that

$$f(z+h) = f(z) \qquad \forall\, z \in \mathcal{H}.$$

Proposition / Definition

The function f admits a Fourier expansion at ∞ of the following form:

$$f(z) = \sum_{n \geq 0} a_f(n)\, e\!\left(\frac{nz}{h}\right),$$

where $a_f(n) = e^{\frac{2\pi n y}{h}} \frac{1}{h} \int_0^h f(u+iy)\, e\!\left(-\frac{nu}{h}\right) du.$

The $a_f(n)$ are called the Fourier coefficients of f. $a_f(0)$ is called the constant term of f.

We formalize some notions we saw in examples:

Proposition If A is the exponent appearing in the growth conditions for f, one has

$$a_f(n) \ll n^{k/2+A}$$

In particular if f is cuspidal then

$$a_f(n) \ll n^{k/2}$$

Proof For $y > 0$ we have

$$|a_f(n)| \ll e^{\frac{2\pi n y}{h}}\, y^{-k/2} (y + y^{-1})^A.$$

we take $y = 1/n$, $\Rightarrow$ $|a_f(n)| \ll e^{2\pi/h}\, n^{k/2+A}.$

Therefore the Fourier expansion of f is rapidly converging and as $y \to \infty$

$$f(z) = a_f(0) + O_{k,A,h}\!\left(e^{-2\pi y/h}\right) \tag{$\star$}$$

In particular, if f is a cusp form, we have $a_f(0) = 0$

since $y^{k/2}|f(x+iy)|$ is bounded as $y\to\infty$ (and $k>0$).

But conversely if $a_f(0)=0$ then $|f(z)| \ll e^{-2\pi y/h}$, so $F(z)$ is bounded.

(**)

Better: $y^{k/2}|f(x+iy)| \ll y^{k/2}\exp(-2\pi h/y) \ll \exp(-\pi h/y)$

So $F(z)$ is decreasing exponentially fast as $y\to\infty$ for cusp forms

Theorem A modular form $f\in M_k(\Gamma)$ $k\in\mathbb{Z}$ is a cusp form if and only if one of the following equivalent conditions is satisfied (setting $F(z)=y^{k/2}|f(z)|$).

(1) $F(z)$ is bounded on $\mathcal{H}$.

(2) For any $\gamma\in GL_2^+(\mathbb{Q})$, the constant term of $f|_\gamma$ is zero. $a_{f|_\gamma}(0)=0$.

(3) For any $\gamma\in GL_2^+(\mathbb{Q})$, $F(\gamma z)$ is decreasing exponentially as $y\to\infty$, i.e. $\exists\, c=c(\gamma)>0$ such that

$$F(\gamma z)\ll \exp(-cy)$$

(4) For any $\gamma\in SL_2\mathbb{Z}$, the constant term of $f|_\gamma$ is zero $a_{f|_\gamma}(0)=0$.

(5) For any $\gamma\in SL_2\mathbb{Z}$, $F(\gamma z)$ is decreasing exponentially fast as $y\to 0$.

Proof We prove

$$\begin{tikzcd} & (1) \arrow[dl, Rightarrow] & \\ (2) \arrow[r, Rightarrow] & (4) \arrow[u, Rightarrow] \arrow[r, Leftrightarrow] & (5) \\ (3) \arrow[u, Leftrightarrow] & & \end{tikzcd}$$

(1) ⇒ (2)

If $F(z)$ bounded on all of $\mathcal{H}$, then $y^{k/2}|f|_\gamma(z)|$ is also bounded for all $\gamma\in GL_2^+(\mathbb{Q})$. so (2)

(2) ⇔ (3) We saw by (*) and (**).

Likewise ④ $\Leftrightarrow$ ⑤.

② $\Rightarrow$ ④ is clear since $SL_2\mathbb{Z} \subseteq GL_2^+(\mathbb{Q})$.

④ $\Rightarrow$ ①. $|F(z)|$ is bounded on $\mathcal{F}_{SL_2\mathbb{Z}}$, since it is continuous, and decaying rapidly into the only non-compact direction.

But then $F(z)$ is bounded on $\gamma\mathcal{F}_{SL_2\mathbb{Z}}$, and since $\mathcal{F}_\Gamma$ is a finite union of $\gamma\mathcal{F}_{SL_2\mathbb{Z}}$'s, $F(z)$ is bounded on $\mathcal{F}_\Gamma$, hence on all of $\mathcal{H}$.

QED

Note that using condition ④ by Γ-invariance, it is sufficient to check that $a_{f|_\gamma}(0)=0$ for γ running through a finite set of representatives for $\Gamma\backslash SL_2\mathbb{Z}$.

Exercise Generalize this proposition to $1/2$-integer weight forms.

In particular:

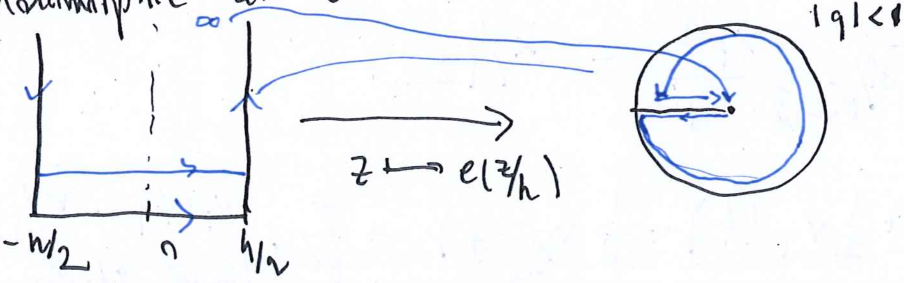
COROLLARY The subspace $S_k(\Gamma)$ of $M_k(\Gamma)$ is of codimension at most $[SL_2\mathbb{Z}:\Gamma]$ in $M_k(\Gamma)$.

PROOF $S_k(\Gamma)$ is the intersection of the kernels of the linear maps $f \mapsto a_{f|_{\gamma_i}}(0)$ for $\gamma_i \in \Gamma\backslash SL_2\mathbb{Z}$.

In particular, to show $\dim M_k(\Gamma)<\infty$, it suffices to show $\dim S_k(\Gamma)<\infty$.

The growth condition is often interpreted as saying that $f\in \mathcal{F}_k(\Gamma)$ is "holomorphic at ∞".

Indeed



But since $f(z+h)$, and f is of moderate growth, we have with $q = e(z/h)$

$$\tilde{f}(q) = \sum_{n \geq 0} a_f(n) q^n$$ defines a holomorphic function in $\mathbb{D} = \{q : |q| < 1\}$

$\tilde{f}$ holomorphic at $q=0$ $\iff$ f holomorphic at ∞ $\iff$ f is of moderate growth.

Next goal: Dimension formula for $SL_2\mathbb{Z}$.

Let $f : \mathcal{H} \to \mathbb{C}$ be a meromorphic function, not identically zero

Let $p \in \mathcal{H}$.

The integer $n \in \mathbb{Z}$ such that $\frac{f(z)}{(z-p)^n}$ is holomorphic and non-zero is called the order of f at p. We write $v_p(f)$.

Suppose f is modular of weight k for $SL_2\mathbb{Z}$,

Then $v_p(f) = v_{\gamma p}(f)$ $\forall \gamma \in SL_2\mathbb{Z}$.

So $v_p(f)$ only depends on the image of p in $\mathcal{H}/PSL_2\mathbb{Z}$.

Let also $v_\infty(f)$ be the order of vanishing of $\tilde{f}(q)$ at 0.

Let $e_p = |\mathrm{Stab}_{PSL_2\mathbb{Z}}\, p|$

Recall from when we constructed the fundamental domain for $SL_2\mathbb{Z}$ that

$$e_p = \begin{cases} 2 & \text{if } p = \gamma i \text{ for some } \gamma \in SL_2\mathbb{Z} \\ 3 & \text{if } p = \gamma \zeta_3 \text{ for some } \gamma \in SL_2\mathbb{Z} \\ 1 & \text{else.} \end{cases}$$

<u>Theorem</u> Let $f \in M_k(SL_2\mathbb{Z})$. We have

$$v_\infty(f) + \sum_{p \in \mathcal{H}/PSL_2\mathbb{Z}} \frac{v_p(f)}{e_p} = k/12$$

Also: $$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_{\zeta_3}(f) + \sum_{\substack{p \in \mathcal{H}/SL_2\mathbb{Z} \\ p \neq i, \zeta_3}} v_p(f) = k/12$$

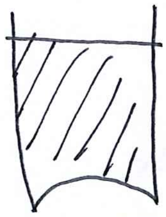
Note that the sum in the theorem makes sense:

f has only a finite number of zeros ~~and/poles~~ mod $SL_2\mathbb{Z}$

$\tilde{f}$ is holomorphic on $\mathbb{D}$: $\exists\, r>0$ such that $\tilde{f}(z)$ has no zeros in

$0<|q|<r$

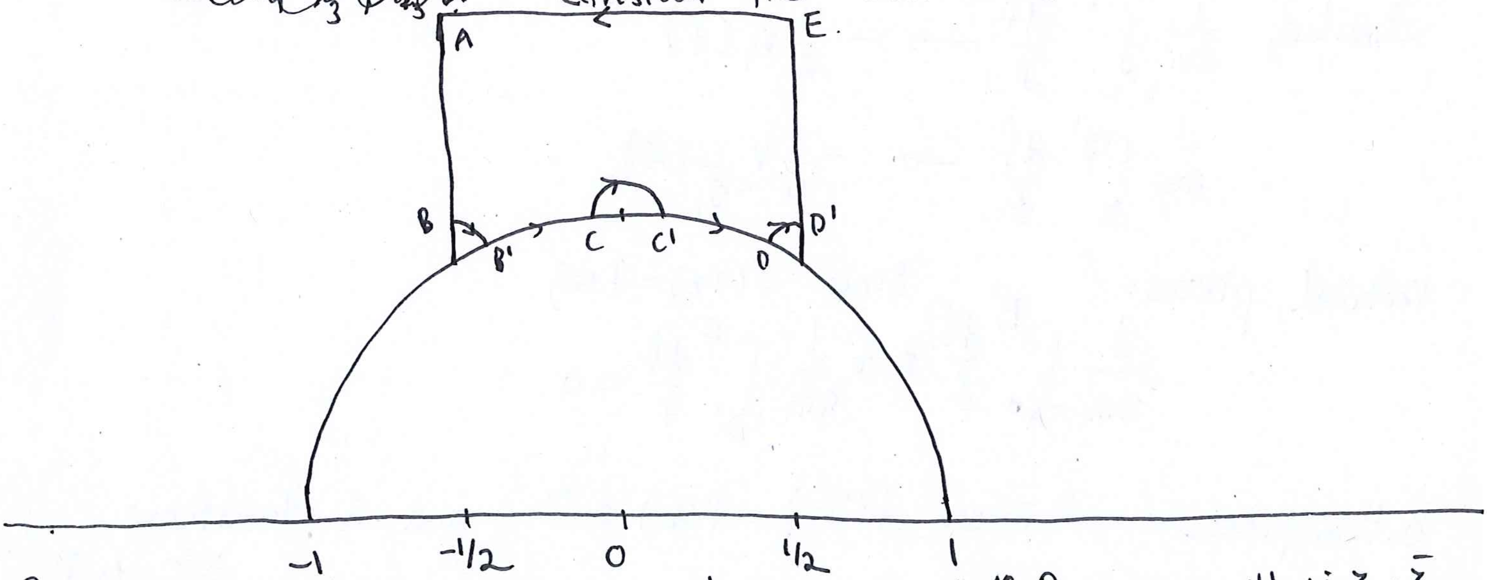
$\iff$ f has no zero for $\mathrm{Im}(z) > \frac{1}{2\pi}\log 1/r$, but



is a compact region: there are only finitely many zeros here.

PROOF Idea: Integrate $\frac{1}{2\pi i}\frac{df}{f}$ on the boundary of $\mathcal{D}$. ~~except possibly~~

Consider the contour $\mathcal{C}$



Suppose that f has no zero on the boundary of $\mathcal{D}$ except possibly at $i, \zeta_3, -\bar{\zeta}_3$.
If the circles around ζ_3, i, $-\bar{\zeta}_3$ and ∞ are sufficiently small, then $\mathcal{C}$ contains every zero of f not congruent to i or ζ_3

Residue Theorem: $\displaystyle \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{df}{f} = \sum_{\substack{p\in \mathcal{H}/PSL_2\mathbb{Z} \\ p\neq i,\zeta_3}} v_p(f)$

On the other hand, we can calculate each piece of the contour individually:

Segment AE: We make the change of variable $q=e(z)$,

$\displaystyle \frac{1}{2\pi i}\int_E^A \frac{df}{f} = \frac{1}{2\pi i}\int_{\omega} \frac{d\tilde{f}}{\tilde{f}}$, where ω is a small circle around the origin in $\mathbb{D}$

with negative orientation, and so small so that it does not ~~contain~~ enclose any zero of $\tilde{f}(q)$, except possibly at $q=0$.

So $$\frac{1}{2\pi i}\int_E^A \frac{df}{f} = -v_\infty(f).$$

Let ω_2 be the circle centered at ζ_3 extending the arc BB'.

Then $$\frac{1}{2\pi i}\int_{\omega_2}\frac{df}{f} = -v_{\zeta_3}(f)$$

When the radius $\to 0$, the angle $B \measuredangle_{\zeta_3} B' \to \frac{2\pi}{6}$, so

$$\frac{1}{2\pi i}\int_B^{B'}\frac{df}{f} = -\frac{1}{6}v_{\zeta_3}(f) \text{ as radius} \to 0.$$

Similarly $$\frac{1}{2\pi i}\int_C^{C'}\frac{df}{f} \to -\frac{1}{2}v_i(f)$$

$$\frac{1}{2\pi i}\int_D^{D'}\frac{df}{f} \to -\frac{1}{6}v_{-\bar{\zeta}_3}(f).$$

Vertical pieces: Since $f(Tz)=f(z)$

$$\frac{1}{2\pi i}\int_A^B\frac{df}{f} + \frac{1}{2\pi i}\int_{D'}^E\frac{df}{f} = 0$$

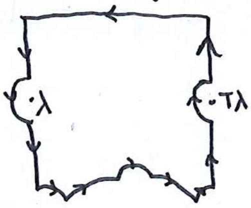
Bottom piece: Since $f(Sz) = f(z)z^k$, and S transforms $B'C$ to DC'

$$\frac{df(Sz)}{f(Sz)} = k\frac{dz}{z} + \frac{df(z)}{f(z)},$$ so that

$$\frac{1}{2\pi i}\left(\int_{B'}^C + \int_{C'}^D\right)\frac{df}{f} = \frac{1}{2\pi i}\int_{B'}^C \frac{df}{f} - \frac{df(Sz)}{f(Sz)} = \frac{1}{2\pi i}\int_{B'}^C\left(-k\frac{dz}{z}\right) \to -k\left(\frac{-1}{12}\right) = \frac{k}{12}$$

Therefore the formula is as claimed.

If f has a zero λ on $\{z : \mathrm{Re}(z) = -1/2,\ \mathrm{Im}\, z > \frac{\sqrt{3}}{2}\}$



The arc circling around $T\lambda$ is the transform of the arc ~~and~~ around λ.

Similarly for zeros on lower boundary

as several zeros

QED

What about the dimensions of the spaces $M_k(SL_2\mathbb{Z})$?

Well, we have $\dim M_k(SL_2\mathbb{Z})/S_k(SL_2\mathbb{Z}) \leq 1$ and for $k \geq 4$ that $E_k \in M_k(SL_2\mathbb{Z}) \setminus S_k(SL_2\mathbb{Z})$

So $\dim M_k/S_k = 1$ and $M_k(SL_2\mathbb{Z}) = S_k(SL_2\mathbb{Z}) \oplus \mathbb{C}\cdot E_k$ $k \geq 4$.

Let $\Delta = \dfrac{E_4^3 - E_6^2}{2^6 3^3} \in S_k(SL_2\mathbb{Z})$ the Ramanujan Δ-fcn.

PROPOSITION ① $M_k(SL_2\mathbb{Z}) = 0$ if $k<0$, or $k=2$ or k odd

② Multiplication by Δ defines an isomorphism $M_{k-12}(SL_2\mathbb{Z}) \xrightarrow{\sim} S_k(SL_2\mathbb{Z})$.

③ For $k = 0, 4, 6, 8, 10$, $M_k(SL_2\mathbb{Z})$ is 1-diml with basis $1, E_4, E_6, E_8, E_{10}$, i.e. $S_k(SL_2\mathbb{Z}) = 0$ for $0, 4, 6, 8, 10$.

PROOF Let $f \in M_k$ Recall $(-1)^k = 1$ else $M_k = 0$, so k even if $M_k \neq 0$.

① All the terms on the Left hand side of

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_{\zeta_3}(f) + \sum_{\substack{p \in \mathcal{H}/PSL_2\mathbb{Z} \\ p \neq i, \zeta_3}} v_p(f) = k/12.$$

are ≥ 0 so $k \geq 0$.

Also $k=2$ is not possible since $1/6$ is not representable as $n_1 + \frac{1}{2}n_2 + \frac{1}{3}n_3$.

② Let $k=4$. Then we must have

$v_{\zeta_3}(E_4) = 1$, $v_i(E_4) = 0$, $v_\infty(E_4) = 0$, $v_p(E_4) = 0$ for other p.

$\Rightarrow E_4$ has a simple zero at $SL_2\mathbb{Z}\cdot\zeta_3$ and no where else

Let $k=6$ Then we must have

$v_{\zeta_3}(E_6) = 0$ $v_i(E_6) = 1$, $v_\infty(E_6) = 0$ $v_p(E_6) = 0$ for other p.

$\Rightarrow E_6$ has a simple pole at $SL_2\mathbb{Z}\cdot i$ and nowhere else.

Δ is weight 12, and we have $v_\infty(\Delta) = 1$, so Δ has no zeros in $\mathcal{H}$.

If $f \in \cancel{M_k} S_k(SL_2\mathbb{Z})$ and $g = f/\Delta$, then g is weight $k-12$ and $v_p(g) = v_p(f) - v_p(\Delta) = \begin{cases} v_p(f) & \text{if } p \neq \infty \\ v_p(f) - 1 & \text{if } p = \infty \end{cases}$

so $v_p(g) \geq 0 \ \forall p \implies g$ is holomorphic, moderate growth, modular.

so $g \in M_{k-12}(SL_2\mathbb{Z})$.

(3) We have $k-12 < 0$ if $k \leq 10$, so $S_k(SL_2\mathbb{Z}) = 0$ in these cases.

Q.E.D.

COROLLARY $k \geq 0$

$$\dim M_k(SL_2\mathbb{Z}) = \begin{cases} \lfloor k/12 \rfloor & k \equiv 2 \ (12) \\ \lfloor k/12 \rfloor + 1 & k \not\equiv 2 \ (12). \end{cases}$$
