Classical Modular Forms Lecture 8 19.10.2017

Recall $\dim M_k(SL_2\mathbb{Z})/S_k(SL_2\mathbb{Z}) = 1$

and we showed if f not identically zero

$$v_\infty(f) + \frac{1}{2} v_i(f) + \frac{1}{3} v_{\zeta_3}(f) + \sum_{\substack{p \in \mathcal{H}/PSL_2\mathbb{Z} \\ p \not\cong i, \zeta_3}} v_p(f) = k/12$$

where $p_1 \cong p_2$ if $\exists\ \gamma \in SL_2\mathbb{Z}$ st $p_1 = \gamma p_2$.

We proved this by an explicit integration around the boundary of $\mathcal{D}_{SL_2\mathbb{Z}}$.

PROPOSITION ① $M_k(SL_2\mathbb{Z}) = 0$ if k odd, or $k<0$, or $k=2$.

② Multiplication by Δ defines an isomorphism $M_{k-12}(SL_2\mathbb{Z}) \to S_k(SL_2\mathbb{Z})$

③ For $k = 0, 4, 6, 8, 10$, $M_k(SL_2\mathbb{Z})$ is 1-diml, with basis $1, E_4, E_6, E_8, E_{10}$

i.e. $S_k(SL_2\mathbb{Z}) = 0$ for these values of k.

PROOF ① Last time

② $\Delta = \dfrac{E_4^3 - E_6^2}{2^6 3^3}$ is weight 12, and $\hat{\Delta}(q)$ vanishes at $q=0$.

~~So~~ But is Δ identically zero?

Well, if $k=4$ we have

$v_{\zeta_3}(E_4) = 1$, $v_i(E_4) = 0$, $v_\infty(E_4) = 0$, $v_p(E_4) = 0$ for other p

Similarly, if $k=6$ we have

$v_{\zeta_3}(E_6) = 0$, $v_i(E_6) = 1$, $v_\infty(E_6) = 0$, $v_p(E_6) = 0$ for other p.

$\Rightarrow$ E_4 has a unique zero at ζ_3, E_6 has a unique zero at i.

so $\Delta(\zeta_3) \neq 0$, $\Delta(i) \neq 0$, in particular $\Delta \not\equiv 0$.

So $v_\infty(\Delta) = 1$, and Δ does not vanish on ~~$\mathcal{H}$~~ $\mathcal{H}$.

Let $f \in S_k(SL_2\mathbb{Z})$, ~~we have $v_\infty(\Delta) = 1$,~~ We have to show

$g = f/\Delta \in M_{k-12}(SL_2\mathbb{Z})$, g is modular of weight $k-12$, holomorphic

And $v_p(g) = v_p(f) - v_p(\Delta) = \begin{cases} v_p(f) & p \neq \infty \\ v_p(f) - 1 & p = \infty \end{cases}$.

So $v_p(g) \geq 0$ $\forall p$, and so $\tilde{g}(q)$ holomorphic at $q=0$ in particular, g is bounded on $\mathcal{F}_{SL_2\mathbb{Z}}$, hence $G(z) = y^{\frac{k-12}{2}}|g(z)|$ is bounded on $\mathcal{H}$ $\Rightarrow g \in M_{k-12}(SL_2\mathbb{Z})$.

③ $k < 12$, $k-12 < 0$ for these values, so $S_k(SL_2\mathbb{Z}) = 0$ by pt ①.

Corollary For $k \geq 0$, k even

$$\dim M_k(SL_2\mathbb{Z}) = \begin{cases} \lfloor \frac{k}{12} \rfloor & k \equiv 2 \ (12) \\ \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \ (12) \end{cases}$$

Proof The formula matches ①, ③ for $0 \leq k \leq 10$, and extends to all $k \geq 12$ by ②, and fact $\dim M_k(SL_2\mathbb{Z}) / S_k(SL_2\mathbb{Z}) = 1$.

The proof of ⊛ was rather clumsy and it would be difficult to generalize to congruence subgroups since ~~the~~ $\mathcal{F}_\Gamma$ can be quite complicated. We need to study hyperbolic geometry more closely.

Hyperbolic Geometery:

Def (The complex projective line)

$$\mathbb{P}^1(\mathbb{C}) = \{ L \subseteq \mathbb{C}^2 ; \ L \ni 0 \text{ a line in } \mathbb{C}^2 \}.$$

Lines passing through the origin are parameterized by their slope:

$z \in \mathbb{C} \cup \{\infty\}$

$L_z : X = zY \qquad z \in \mathbb{C}$

$L_\infty : 0 = Y$, so we identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$.

$GL_2\mathbb{C}$ acts linearly on $\mathbb{C}^2$, ~~so~~

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} aX+bY \\ cX+dY \end{pmatrix},$$

~~So~~ and this action preserves lines through $0 \in \mathbb{C}^2$, so $GL_2\mathbb{C}$ acts on $\mathbb{P}^1(\mathbb{C})$. In terms of slope parameterization:

$$g(L_z) = L_{g.z} \quad \text{with} \quad g.z = \frac{az+b}{cz+d}.$$

Convention: if $c \neq 0$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\infty = \frac{a}{c}$, $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\infty = \infty$

The scalars $Z(\mathbb{C}) := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{C}^* \right\}$ act trivially

So the action is via $PGL_2\mathbb{C} := GL_2\mathbb{C}/Z(\mathbb{C})$,

or better $PSL_2\mathbb{C} = SL_2\mathbb{C}/\pm Id$, since we can normalize a matrix to be $\det = 1$.

Orbits: Let $GL_2\mathbb{C}\backslash\mathbb{P}^1(\mathbb{C})$ be the space of orbits.

This isn't very interesting: there is only one orbit: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\infty = \frac{a}{c}$,

so $\mathbb{P}^1(\mathbb{C}) = GL_2\mathbb{C}.\infty$.

Stabilizer of ∞? $GL_2(\mathbb{C})_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in GL_2\mathbb{C} \right\} =: B(\mathbb{C})$.

"Borel Subgroup"

~~Also have~~ So that $\mathbb{P}^1(\mathbb{C}) \simeq GL_2(\mathbb{C})/B(\mathbb{C})$, with inverse $gB(\mathbb{C}) \mapsto g.\infty$.

Fixed points Given $g \in GL_2\mathbb{C}$, a fixed point of g is a point such that

$g.z = z$.

These are the solutions of $\frac{az+b}{cz+d} = z \iff c^2z + (d-a)z - b = 0$

So either $\mathbb{P}^1(\mathbb{C})$ is fixed by g, if $g \in Z(\mathbb{C})$, or g has at most 2 fixed pts.

In terms of lines: L_z is a fixed point if and only if L_z is an eigenspace for $g \in GL_2\mathbb{C}$ $\quad g: \mathbb{C}^2 \to \mathbb{C}^2$.

We also have the "Unipotent Group" $N(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \right\}$.
and $A(\mathbb{C}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times \right\}$.

Action on lines.

Definition A line in $\mathbb{P}^1(\mathbb{C})$ is either $L \cup \{\infty\}$, where L is a line in $\mathbb{C} \simeq \mathbb{R}^2$, or a circle in $\mathbb{C}$.

PROP 1 We won't have time to prove these, but:
Fractional Linear Transformations preserve lines.

This depends on:

PROP 2 (Bruhat Decomposition)

$$GL_2(\mathbb{C}) = B(\mathbb{C}) \sqcup N(\mathbb{C})\, w\, B(\mathbb{C}),$$

where $w = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

In particular, the Bruhat decomposition is unique, and we also have

$$SL_2\mathbb{C} = N(\mathbb{C})A(\mathbb{C}) \sqcup N(\mathbb{C})\, w\, N(\mathbb{C})A(\mathbb{C}).$$

[Exercise !]

In particular, $GL_2\mathbb{C}$ is generated by w, $Z(\mathbb{C})$, $A(\mathbb{C})$, $N(\mathbb{C})$,
[Exercise, show this implies prop 1.]
[Exercise: ~~to prove the 2 first propositions~~
The Bruhat decomposition is still valid if $\mathbb{C}$ is replaced by a subfield $K \subseteq \mathbb{C}$].

Fractional linear transformations of $GL_2\mathbb{R}^+$.

$GL_2^+\mathbb{R}$ has 3 orbits on $\mathbb{P}^1(\mathbb{C})$.

$\mathcal{H}$, $\overline{\mathcal{H}}$ (complex conjugate) and $\mathbb{P}^1(\mathbb{R}) :=$ lines in $\mathbb{C}^2$ with real slope

$= \mathbb{R} \cup \{\infty\} \subseteq \mathbb{P}^1(\mathbb{C})$.

Note $GL_2^+\mathbb{R}$ acts transitively on each.

We work with $SL_2\mathbb{R}$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = \frac{a}{c}$.

Stabilizer of $\infty \in \mathbb{P}^1(\mathbb{R})$

$$SL_2(\mathbb{R})_\infty = B(\mathbb{R}) \cap SL_2\mathbb{R} = B^1(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : \det g = 1 \right\}$$

Stabilizer of $x \in \mathbb{P}^1(\mathbb{R})$ is a conjugate of ∞ by matrix such that $\sigma_x \infty = x$. A "scaling matrix at x".

$SL_2(\mathbb{R})$ acts transitively on $\mathcal{H}$, in fact, $B^1(\mathbb{R})$ does:

let $z = x + iy$

$$\begin{pmatrix} y^{1/2} & x/y^{1/2} \\ 0 & y^{-1/2} \end{pmatrix} \in B^1(\mathbb{R})$$

$$\begin{pmatrix} y^{1/2} & x/y^{1/2} \\ 0 & y^{-1/2} \end{pmatrix} i = z$$

Stabilizer of i: $SL_2(\mathbb{R})_i = SO_2(\mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [-\pi, \pi] \right\}$.

Stabilizer of $z \in \mathcal{H}$ is conjugate to $SO_2(\mathbb{R})$ by $\begin{pmatrix} y^{1/2} & x/y^{1/2} \\ 0 & y^{-1/2} \end{pmatrix}$.

Matrices of $GL_2\mathbb{R}$ with negative determinant map $\mathcal{H}$ to $\overline{\mathcal{H}}$.

So $\mathbb{P}^1(\mathbb{R}) = SL_2\mathbb{R}.\infty = SL_2\mathbb{R}/B^1(\mathbb{R})$

$\mathcal{H} = SL_2\mathbb{R}.i = SL_2(\mathbb{R})/SL_2(\mathbb{R})_i = SL_2\mathbb{R}/SO_2(\mathbb{R})$.

Fixed points: The fractional linear transformations $\in SL_2\mathbb{R}$ are classified by their fixed points. ~~the fixed points $\in \mathbb{P}^1(\mathbb{C})$ are invariant by conjugation~~. (does not change the eigenspaces)

So we classify elements of $SL_2\mathbb{R}$ as follows:

- $I\!\!\;d$: Fixed points are all of $\mathbb{P}^1(\mathbb{C})$.
- Parabolic: $|\operatorname{tr}(g)| = 2$, has exactly one fixed point and that fixed point is $\in \mathbb{P}^1(\mathbb{R})$.

conjugate to some $n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, $x \in \mathbb{R}$ $\in N(\mathbb{R})$

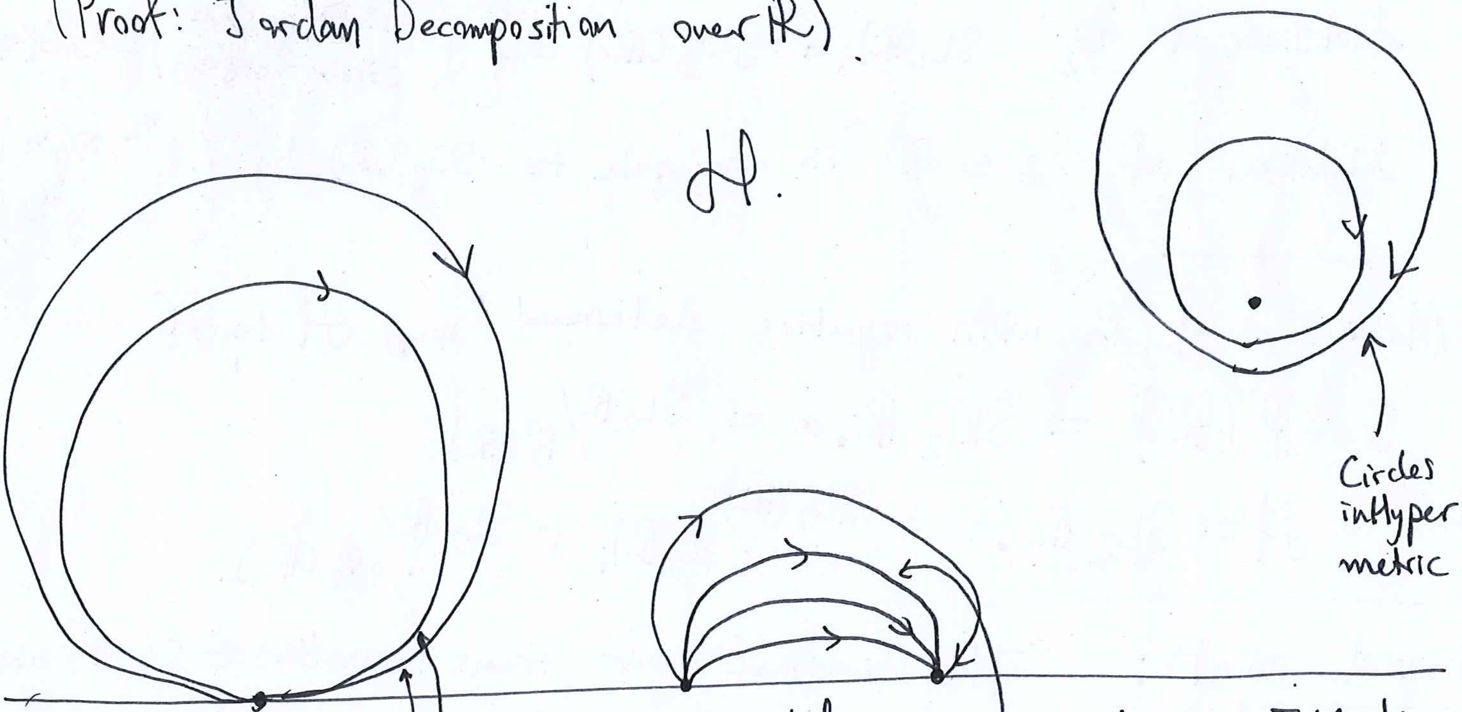
- Hyperbolic: $|\operatorname{tr}(g)| > 2$. Has <u>two</u> fixed points, both on $\mathbb{P}^1(\mathbb{R})$

conjugate to some $a(y) = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}$ $y \in \mathbb{R}_{>0}$ $\in A(\mathbb{R})$

- Elliptic: $|\operatorname{tr}(g)| < 2$, has two fixed points, one in $\mathcal{H}$ and the other in $\overline{\mathcal{H}}$.

conjugate to $g = k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\theta \in [-\pi, \pi]$.

(Proof: Jordan Decomposition over $\mathbb{R}$).



Circles in Hyperbolic metric

Parabolic "Horocycles"

Hyperbolic. a geodesic Elliptic

a horizontal line is also a horocycle: it touches $\mathbb{P}^1(\mathbb{R})$ at one point: ∞.

Only the circle ~~with~~ perpendicular to $\mathbb{P}^1(\mathbb{R})$ is fixed (but not pointwise)

Action on Half-lines :

$SL_2\mathbb{R} \leq GL_2\mathbb{C}$, so sends lines to lines in $\mathbb{P}^1(\mathbb{C})$.

Restrict these lines to half-lines in $\mathcal{H}$:

- If $C \subseteq \mathcal{H}$ is a circle, then $g \cdot \mathcal{H}$ is still a circle $\subseteq \mathcal{H}$.
- If C is horizontal line, or circle in $\mathcal{H}$ tangent to $\mathbb{R}$ (horocycles), then $g.C$ is also a horocycle tangent to $\mathbb{P}^1(\mathbb{R})$ at $g.x$ (i.e. a horizontal line if $g.x = \infty$).
- If L is the restriction to $\mathcal{H}$ of a line in $\mathbb{P}^1(\mathbb{C})$ meeting $\mathbb{P}^1(\mathbb{R})$ in two distinct points x, x' (i.e. L is a non-horizontal half line, or $C \cap \mathcal{H}$) then $g \cdot L$ is the restriction to $\mathcal{H}$ of a line in $\mathbb{P}^1(\mathbb{C})$ meeting $\mathbb{P}^1(\mathbb{R})$ in $g \cdot x$ and $g \cdot x'$.

The hyperbolic metric and measure:

Invariant metric and measure exists on $\mathcal{H}$, and is unique up to multiplied by a constant.

For a differentiable function $f(z)$ on $\mathcal{H}$ we define the differential df by
$$df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial y}{\partial x}\right)dy.$$

~~For $g \in GL_2^+(\mathbb{R})$,~~

We have $dz = dx + i\,dy$, $d\bar z = dx - i\,dy$

$g \in GL_2^+(\mathbb{R})$
$$d(gz) = \left(\frac{d(gz)}{dz}\right) dz = \det g \frac{dz}{(cz+d)^2}$$
$$d(g\bar z) = \frac{d(g\bar z)}{d\bar z}\, d\bar z = \det g \frac{d\bar z}{(c\bar z+d)^2}$$

then we define the metric and measure by

$$ds^2(z) := \frac{dx^2+dy^2}{y^2} = \frac{dz\,d\bar{z}}{y^2}, \quad \text{and} \quad d\mu(z) = \frac{dx\,dy}{y^2}$$

Since $\operatorname{Im}(gz) = \det g \dfrac{\operatorname{Im} y}{|cz+d|^2}$, we have

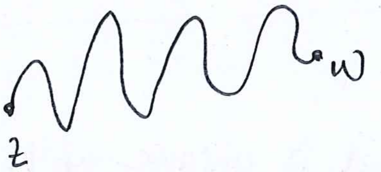
$$\frac{1}{\operatorname{Im}(gz)^2}\, d(gz)\, d(g\bar{z}) = \frac{|cz+d|^4}{y^2 (\det g)^2} (\det g)^2 \frac{dz\,d\bar{z}}{(cz+d)^2(c\bar{z}+d)^2}$$

$$= \frac{dz\,d\bar{z}}{y^2}$$

So the metric is $GL_2^+(\mathbb{R})$-invariant. (similarly the measure)

Called the hyperbolic or Poincaré metric.

Concretely If $L \in C^\infty([0,1] \longrightarrow \mathcal{H})$ with $L(0) = z$, $L(1) = w$.



define its length $l_h(L) := \displaystyle\int_0^1 \frac{\left((x'(t))^2 + (y'(t))^2\right)^{1/2}}{y(t)}\, dt$

$$= \int_0^1 ds(\phi(t)).$$

$l_h(gL) = l_h(L)$.

Defn $\quad d_h(z,z') = \inf_{L} l_h(L) \quad$ st $L(0) = z$, $L(1) = z'$.

Thus $d_h(gz, gz') = d_h(z,z') \quad \forall g \in GL_2^+\mathbb{R}$.

$GL_2^+(\mathbb{R})$ preserves hyperbolic distance.

More on spheres and balls WRT d_h next time.