Recall: lines in $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ are Euclidean lines and circles. $SL_2\mathbb{R} \subseteq GL_2\mathbb{C}$ sends lines to lines.

Restrict these lines to $\mathcal{H}$:

FACTS: $g \in SL_2\mathbb{R}$.

- If $C \subseteq \mathcal{H}$ is a Euclidean circle, then $g.C$ is also a circle $\subseteq \mathcal{H}$.
- If C is a horocycle (a circle tangent to $\mathbb{P}^1(\mathbb{R})$) touching $\mathbb{P}^1(\mathbb{R})$ at exactly one point x, then gC is a horocycle touching $\mathbb{P}^1(\mathbb{R})$ at $g.x$.
- If L is the restriction to $\mathcal{H}$ of a line in $\mathbb{P}^1(\mathbb{C})$ meeting $\mathbb{P}^1(\mathbb{R})$ in two distinct points x, x', then $g.L$ is the restriction to $\mathcal{H}$ of a hyperbolic line meeting $\mathbb{P}^1(\mathbb{R})$ in $g.x$ and $g.x'$.

Hyperbolic metric and measure.

Recall Facts from exercises..

For a differentiable function $f(z)$ on $\mathcal{H}$, define the differential df by $df = \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy$. E.g. $dz = dx + i dy$, $d\bar{z} = dx - i dy$.

a section of $T^*\mathcal{H}$.

We define a symmetric 2-form (i.e. a metric) by

$$ds^2(z) = \frac{(dx)^2 + (dy)^2}{y^2} = \frac{(dz)(d\bar{z})}{y^2} \qquad \left(\text{a section of } \operatorname{Sym}^2 T^*\mathcal{H}\right)$$

Exercise 3.5.1: $ds^2(z)$ is $GL_2^+(\mathbb{R})$-invariant

Called the Hyperbolic or Poincaré metric.

Concretely: If $L \in C^\infty([0,1] \to \mathcal{H})$ with $L(0) = z$, $L(1) = w$

then this defines a segment of a 1-manifold $\subseteq \mathcal{H}$:

$$\mathrm{Re}(L(t)) = x(t)$$
$$\mathrm{Im}(L(t)) = y(t)$$

The length is:
$$\ell_h(L) = \int_0^1 \frac{\left((x'(t))^2 + (y'(t))^2\right)^{1/2}}{y(t)}\, dt$$
$$= \int_0^1 ds(L(t))$$

Definition $$d_h(z,z') = \inf_L \{\ell_h(L) : L(0) = z,\ L(1) = z'\}.$$

Thus: $d_h(z,z') = d_h(gz, gz') \quad \forall g \in GL_2^+(\mathbb{R})$.

$GL_2^+(\mathbb{R})$ preserves the hyperbolic distance

$PGL_2\mathbb{R}$ is the group of orientation-preserving isometries of $\mathcal{H}$.

This d_h defines a metric on $\mathcal{H}$, thus a topology on $\mathcal{H}$ (same as Euclidean topology).

Let $S_h(z,r) = \{z' \in \mathcal{H} : d_h(z,z') = r\}$ sphere

$D_h(z,r) = \{z' \in \mathcal{H} : d_h(z,z') \leq r\}$. disk.

~~Exercise 3.5.1~~: ~~A~~.

Definition: A geodesic between z, z' is an L attaining the inf in the definition of $d_h(z,z')$.

Exercise 3.5.2: A geodesic joining $z = x+iy$ and $z' = x+iy'$ is a vertical segment. It has length

$$d_h(z,z') = \left| \int_y^{y'} \frac{dt}{t} \right| = |\log y/y'|.$$

Eg. Thus $d_h(z, x+iye^{\pm r}) = r$

Exercise Geodesics in $\mathcal{H}$ are lines perpendicular to $\mathbb{P}^1(\mathbb{R})$ at two points.

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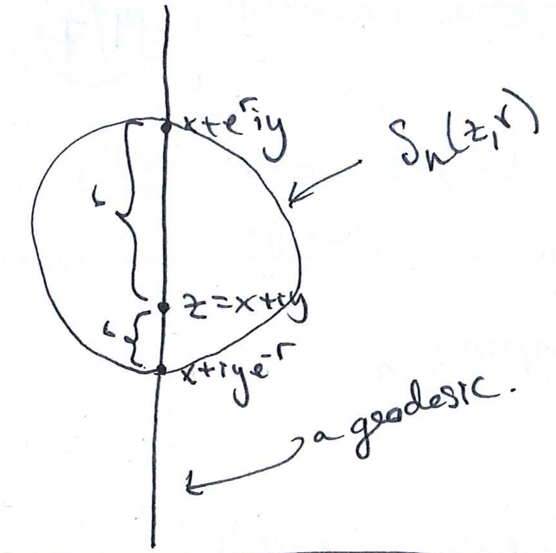
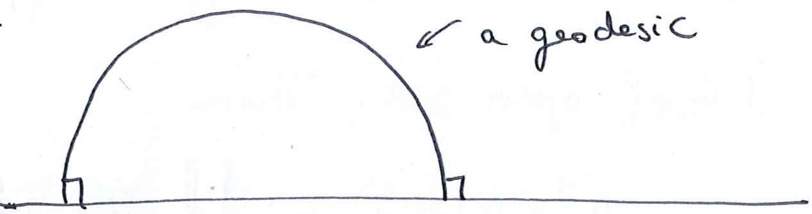




Prop (Proved in exercises)

The geodesic segment joining two points $z \neq z'$ is unique and is a line perpendicular to $\mathbb{P}^1(\mathbb{R})$. Moreover

$$\cosh\left(d_h(z,z')\right) = 1 + 2\,\frac{|z-z'|^2}{4yy'}$$

← Euclidean absolute value.

Hyperbolic measure.

Let $\dfrac{dz \wedge d\bar{z}}{y^2} = 2i\,\dfrac{dx \wedge dy}{y^2}$ be an alternating 2-form.

It is called the hyperbolic measure.

Classically: $d\mu(z) = \cancel{dx \wedge dy}\ \dfrac{dx\,dy}{y^2}$

It is $GL_2^+\mathbb{R}$-invariant.

$$\int_{\mathcal{H}} f(z)\,\frac{dx\,dy}{y^2} = \int_{\mathcal{H}} f(gz)\,\frac{dx\,dy}{y^2}$$

$\forall g \in GL_2^+\mathbb{R}$ for any f such that the $\int$ converges absolutely.

~~Let~~ Geometry of Modular Curves: $\Gamma \subseteq SL_2\mathbb{Z}$ finite index.

Let $Y(\Gamma) = \{\Gamma z : z \in \mathcal{H}\}$ space of orbits

Let $\pi : \mathcal{H} \to Y(\Gamma)$, $\pi(z) = \Gamma(z)$ be the canonical projection.

We define structures on $Y(\Gamma)$

So that this map is continuous, isometric, holomorphic.

Defn A subset $\mathcal{U} \subseteq Y(\Gamma)$ is open if $\pi^{-1}(\mathcal{U})$ is open in $\mathcal{H}$.

This is the finest topology in which π is continuous.

Note that π is also an open mapping:

Let $\mathcal{U} \subseteq \mathcal{H}$ open set. Then

$$\pi^{-1}(\pi(\mathcal{U})) = \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{U}) =: \Gamma(\mathcal{U})$$

which is also open. Thus by definition $\pi(\mathcal{U})$ is open.

Exercise (a) $\pi(\mathcal{U}_1) \cap \pi(\mathcal{U}_2) = \emptyset$ in $Y(\Gamma)$

if and only if

$\Gamma(\mathcal{U}_1) \cap \mathcal{U}_2 = \emptyset$ in $\mathcal{H}$

(b) $Y(\Gamma)$ is connected.

We want to show that $Y(\Gamma)$ is Hausdorff.

Key step is to show that any two points which are distinct also have small enough neighborhoods so that every $SL_2\mathbb{Z}$ ~~trans~~ translate of the neighborhoods is disjoint.

That is to say: The action of $SL_2\mathbb{Z}$ on $\mathcal{H}$ is properly discontinuous.

Precisely: PROPOSITION:

Let $z_1, z_2 \in \mathcal{H}$ then there exists $\mathcal{U}_1 \ni z_1$, $\mathcal{U}_2 \ni z_2$, $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{H}$ such that

For all $\gamma \in SL_2\mathbb{Z}$ if $\gamma(\mathcal{U}_1) \cap \mathcal{U}_2 \neq \emptyset$ then $\gamma z_1 = z_2$

(If 

then can shrink 

. If we cant shrink, then $\gamma z_1 = z_2$)

PROOF Let U_1' a neighborhood of z_1 in $\mathcal{H}$ with compact closure

similarly U_2'.

Let $\gamma \in SL_2\mathbb{Z}$ and consider $\gamma(U_1') \cap U_2'$.

Claim For all but finitely many pairs (c,d) of integers such that $\gcd(c,d)=1$, we have

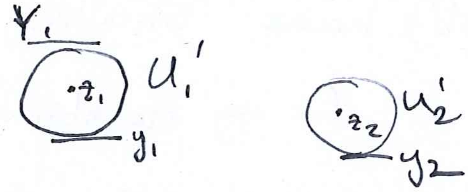
$$\sup\left\{ \operatorname{Im} \gamma z : \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in SL_2\mathbb{Z},\ z \in U_1' \right\} \le \inf\{ \operatorname{Im} z : z \in U_2' \}.$$

PROOF of Claim

Let: $y_1 = \inf\{ \operatorname{Im} z : z \in U_1' \}$

$Y_1 = \sup\{ \operatorname{Im} z : z \in U_1' \}$

$y_2 = \inf\{ \operatorname{Im} z : z \in U_2' \}$



Then if $z \in U_1'$ $\quad \operatorname{Im} \gamma z = \dfrac{\operatorname{Im} z}{|cz+d|^2}$

$$\Rightarrow \operatorname{Im}(\gamma z) \le \min\left(\frac{1}{c^2 y_1}, \frac{Y_1}{(cx+d)^2} \right)$$

But $\dfrac{1}{c^2 y_1} \le y_2$ for all but finitely many $c \in \mathbb{Z}$ $\left(c^2 \le \dfrac{1}{y_1 y_2} \right)$.

For each of these exceptional c

$\dfrac{Y_1}{(cx+d)^2} \le y_2$ for all but finitely many d (uniformly in z).

Thus there are only finitely many (c,d) pairs for which $\operatorname{Im} \gamma z \le y_2$ does not hold.

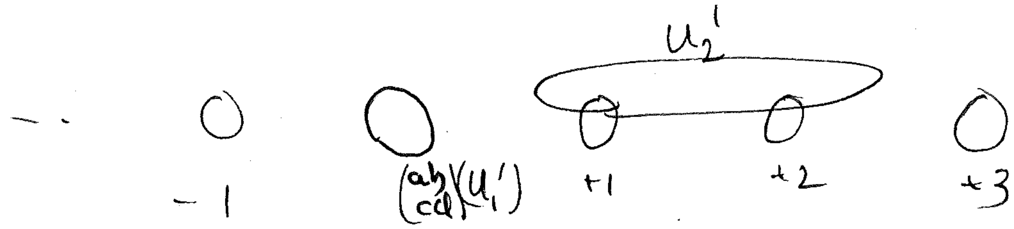
By the claim, $\gamma(U_1') \cap U_2'$ is empty for all but finitely many orbits $\gamma \in \Gamma_\infty \backslash SL_2\mathbb{Z}$

On the other hand for each $\tilde{\gamma} \in \Gamma_\infty \backslash SL_2\mathbb{Z}$

Thus for all but finitely many $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\gamma}$

$$\gamma(U_1') \cap U_2' = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} (U_1') + k \right) \cap U_2' \text{ is empty}, \quad k \in \mathbb{Z}$$

(U_2' has compact closure)



Therefore $\gamma(U_1') \cap U_2'$ for only finitely many $\gamma \in SL_2\mathbb{Z}$.

We have shown:

For any neighborhoods with compact closure $U_1 \ni z_1$, $U_2 \ni z_2$.

$$|\{\gamma \in SL_2\mathbb{Z} : \gamma U_1 \cap U_2\}| < \infty.$$

Let $F = \{\gamma \in SL_2\mathbb{Z} : \gamma(U_1') \cap U_2' \neq \emptyset \text{ and } \gamma z_1 \neq z_2\}$

F is a finite set.

For each $\gamma \in F$ there are disjoint neighborhoods $U_{1,\gamma} \ni z_1$, $U_{2,\gamma} \ni z_2$.

Define $U_1 = U_1' \cap \left(\bigcap_{\gamma \in F} \gamma^{-1}(U_{1,\gamma}) \right)$ a nbhd of z_1

$U_2 = U_2' \cap \left(\bigcap_{\gamma \in F} U_{2,\gamma} \right)$ a nbhd of z_2.

These U_1, U_2 satisfy the proposition. Indeed:

Let $\gamma \in SL_2\mathbb{Z}$ such that $\gamma(U_1) \cap (U_2) \neq \emptyset$

To show $\gamma z_1 = z_2$ it suffices to show $\gamma \notin F$.

Suppose $\gamma \in F$. $\gamma^{-1}(U_{1,\gamma}) \supseteq U_1$, $U_{2,\gamma} \supseteq U_2$ so $U_{1,\gamma} \cap U_{2,\gamma} \supseteq \gamma U_1 \cap U_2 \neq \emptyset$ CONTRADICTION.

COROLLARY There exists $\mathcal{U} \ni z$ so that

$$\{\gamma \in SL_2\mathbb{Z} : \gamma(\mathcal{U}) \cap \mathcal{U} \neq \emptyset\} = (SL_2\mathbb{Z})_z .$$

COROLLARY

For any finite-index subgroup $\Gamma \subseteq SL_2\mathbb{Z}$, $Y(\Gamma)$ is Hausdorff.

PROOF Let $\pi(z_1)$, $\pi(z_2)$ disinct points in $Y(\Gamma)$.

Let $\mathcal{U}_1 \ni z_1$, $\mathcal{U}_2 \ni z_2$ as in the previous proposition.

Since $\gamma(z_1) \neq z_2$ $\forall \gamma \in \Gamma$ we have

$\Gamma(\mathcal{U}_1) \cap \mathcal{U}_2 = \emptyset$ in $\mathcal{H}$ by the prop.

So exercise shows that $\pi(\mathcal{U}_1)$ and $\pi(\mathcal{U}_2)$ are disjoint sets in $Y(\Gamma)$, $\pi(\mathcal{U}_1) \ni \pi(z_1)$ $\pi(\mathcal{U}_2) \ni \pi(z_2)$.

Since π is an open mapping, they are neighborhoods.

~~Also by~~ Cor

Lemma: For any finite-index subgroup $\Gamma \leq SL_2\mathbb{Z}$, $Y(\Gamma)$ is locally compact.

PROOF By the open mapping property, a basis of ~~neigh~~ neighborhoods of a pt $z \in \mathcal{H}$ are mapped by π to a basis of nbhds for $\pi(z) \in Y(\Gamma)$.

Thus π maps compacts to compacts.

Thus $Y(\Gamma)$ is locally compact.

Note also the stabilizer Γ_z of a point is finite

Since $\Gamma_z \leq SL_2\mathbb{Z}_z = g\, SO_2\, g^{-1} \cap SL_2\mathbb{Z}$ . .