

## MODULAR FORMS EXERCISES AND SOLUTIONS

### 1. DUE ON 26TH SEPTEMBER

1.1. **Exercise.** Let  $\mathcal{P}$  be the set of primes. Prove that  $\sum_{p \in \mathcal{P}} \frac{1}{p} = +\infty$ .

1.2. **Solution.** Let  $s > 1$ . Then from the Euler product of the Zeta function,

$$\begin{aligned} \log \zeta(s) &= \sum_{p \in \mathcal{P}} -\log(1 - p^{-s}) = \sum_{p \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} \\ &\leq \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \sum_{k=2}^{\infty} \frac{1}{p^k} = \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \frac{1}{p(p-1)} \\ &= \sum_{p \in \mathcal{P}} \frac{1}{p^s} + O(1) \end{aligned}$$

As we know that  $\lim_{s \rightarrow 1+} \zeta(s) = +\infty$ , letting  $s \rightarrow 1+$  in the above inequality we conclude that

$$\lim_{s \rightarrow 1+} \sum_{p \in \mathcal{P}} \frac{1}{p^s} = +\infty,$$

hence the result.

1.3. **Summation by Parts.** Let  $a : \mathbb{N} \rightarrow \mathbb{C}$  be an arithmetic function, let  $0 < y < x$  and let  $f : [y, x] \rightarrow \mathbb{C}$  be a function with continuous derivative on  $[y, x]$ . Then

$$\sum_{y < n \leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where  $A(x) = \sum_{n \leq x} a_n$ .

1.4. **Exercise.** Prove that for every  $\delta > 0$ ,

$$\pi(x) := |\{p \in \mathcal{P} \mid p \leq x\}|$$

is bigger than  $\frac{x}{(\log x)^{1+\delta}}$  for some sufficiently large  $x$ .

1.5. **Solution.** Let  $a_n$  be the prime indicator function, i.e.

$$a_n := \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not a prime.} \end{cases} .$$

Using summation by parts we note that,

$$\sum_{p \leq x} \frac{1}{p} = \sum_{3/2 < n \leq x} \frac{a_n}{n} = \frac{\pi(x)}{x} + \int_{3/2}^x \frac{\pi(t)}{t^2} dt.$$

If the claim is false i.e. for all sufficiently large  $x$ ,  $\pi(x) \leq x/(\log x)^{1+\delta}$  then from the above,

$$\sum_{p \leq x} \frac{1}{p} \leq \frac{1}{(\log x)^{1+\delta}} + C + \frac{1}{(\log x)^\delta},$$

for some constant  $C$ . The RHS of the above tends to  $C$  as  $x \rightarrow \infty$  contradicting Exercise 1.1, hence the result.

1.6. **Exercise.** Prove that for  $\Re(s) > 1$ ,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx,$$

where  $\{x\}$  is the fractional part of  $x$ . Using this show that  $\zeta(s)$  has meromorphic continuation to  $\Re(s) > 0$  with a simple pole at  $s = 1$ .

1.7. **Solution.** Let  $\Re(s) > 1$ . Then using the summation by parts as following.

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n^s} &= \frac{[x]}{x^s} + s \int_1^x \frac{[t]}{t^{s+1}} dt = \frac{1}{x^{s-1}} - \frac{\{x\}}{x^s} + s \int_1^x t^{-s} dt - s \int_1^x \frac{\{t\}}{t^{s+1}} dt \\ &= \frac{s}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}} dt + O(x^{-\Re(s)} + x^{-\Re(s)+1}). \end{aligned}$$

Letting  $x \rightarrow \infty$ , as  $\Re(s) > 1$ , we conclude that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$

We now note that the integral right hand side is well defined for  $\Re(s) > 0$  and is holomorphic in  $s$ . As  $\frac{s}{s-1}$  is a meromorphic function with simple pole at  $s = 1$  and residue 1, we conclude the meromorphic continuation of  $\zeta(s)$  to  $\Re(s) > 0$ .

1.8. **Exercise.** Prove that the Gamma function, which is defined for  $\Re(s) > 0$  by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt$$

has analytic continuation to  $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$  with simple pole at each non-positive integer. Find the residues of the Gamma function at those poles.

**Hint:** First prove that  $\Gamma(s+1) = s\Gamma(s)$ .

1.9. **Solution.** By integration by parts we see that

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1} \frac{dt}{t} = \int_0^\infty e^{-t} s t^s \frac{dt}{t} = s\Gamma(s),$$

for  $\Re(s) > 0$ . Thus  $\Gamma(s) = \frac{\Gamma(s+1)}{s}$  extends definition of  $\Gamma(s)$  to  $\Re(s) > -1$  meromorphically with pole at  $s = 0$  as

$$\lim_{s \rightarrow 0^+} \int_0^\infty e^{-t} t^s \frac{dt}{t} = +\infty.$$

The pole is simple, as  $\lim_{s \rightarrow 0} s\Gamma(s) = 1$ , and with residue 1. Similarly  $\Gamma(s)$  can be extended to all  $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$  with simple poles at  $s = -n$ ,  $n \in \mathbb{N}$  with residue,

$$\lim_{s \rightarrow -n} (s+n)\Gamma(s) = \lim_{s \rightarrow -n} \frac{\Gamma(s+n+1)}{(s+n-1)\dots s} = \frac{(-1)^n}{n!}.$$

1.10. **Exercise.** Prove the Poisson summation formula: Let  $f \in \mathcal{S}(\mathbb{R})$  be in the Schwartz class. Prove that

$$\sum_{n \in \mathbb{Z}} f(n+u) = \sum_{n \in \mathbb{Z}} \hat{f}(n)e(nu).$$

**Note:** Putting  $u = 0$  we get the usual Poisson summation formula.

1.11. **Solution.** Let

$$F(x) := \sum_{n \in \mathbb{Z}} f(n+x)$$

which is a function on  $L^1(\mathbb{R}/\mathbb{Z})$  so has a Fourier expansion of the form

$$F(x) = \sum_{n \in \mathbb{Z}} e(nx)\hat{F}(n).$$

Here

$$\begin{aligned} \hat{F}(n) &= \int_0^1 F(x)e(-nx)dx = \sum_{n \in \mathbb{Z}} \int_0^1 \sum_{m \in \mathbb{Z}} f(m+x)e(-nx)dx \\ &= \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(x)e(-nx)dx = \int_{-\infty}^\infty f(x)e(-nx)dx = \hat{f}(n), \end{aligned}$$

this provides the result.

1.12. **Exercise.** Recall that,

$$G(1, N) := \sum_{n \pmod N} e(n^2/N).$$

Prove that

- (1) For any odd positive integer  $N$ ,  $G(1, N^2) = N$  and  $G(1, N^3) = NG(1, N)$ .
- (2) For every positive integer  $N$ ,  $G(1, N) = \frac{1+i^{-N}}{1-i} \sqrt{N}$ .

1.13. **Solution.** (1) is elementary. We can parametrize the residue class of  $N^k$  by

$$\{a_1 N^{k-1} + a_2 N^{k-2} + \cdots + a_k \mid 0 \leq a_i \leq N-1\}.$$

Using this we have,

$$\begin{aligned} G(1, N^2) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(aN+b)^2}{N^2}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \sum_{a=0}^{N-1} e\left(\frac{2ab}{N}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \delta_{b=0} N = N. \end{aligned}$$

Similarly,

$$\begin{aligned} G(1, N^3) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{c=0}^{N-1} e\left(\frac{(aN^2 + bN + c)^2}{N^3}\right) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) \sum_{a=0}^{N-1} e(2ac/N) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) N \delta_{c=0} = NG(1, N). \end{aligned}$$

For the second part we use the Poisson summation formula. First we note the function

$$f(x) := 1_{[0, N]} e(x^2/N)$$

is a function which is continuous on  $(0, N)$  and has continuity only from one side at  $x = 0, N$ . From the Fourier theory we know that the Fourier series of  $f$  at  $x = 0$  would converge to  $\frac{f(0+) + f(0-)}{2} = f(0+)/2$ . and similarly, at  $x = N$  to  $f(N-)/2$ . Thus using the (modified) Poisson summation formula and using that  $f(0+) = f(N-)$  we get that,

$$\begin{aligned} \sum_{n=0}^N e(N^2/N) &= \frac{f(0+)}{2} + \sum_{n=1}^{N-1} f(n) + \frac{f(N-)}{2} \\ &= \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(nx) dx = \sum_{n \in \mathbb{Z}} \int_0^N e(x^2/N + nx) dx. \end{aligned}$$

Thus,

$$G(1, N) = N \sum_{n \in \mathbb{Z}} \int_0^1 e(Nx^2 + nNx) dx = N \sum_{n \in \mathbb{Z}} e(-Nn^2/4) \int_0^1 e(N(x + n/2)^2)$$

Noting that

$$e(-Nn^2/4) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ i^{-N}, & \text{if } n \text{ is odd.} \end{cases}$$

and dividing the above sum into odd and even parts we get that,

$$\begin{aligned} G(1, N) &= N \sum_{n \in \mathbb{Z}} \int_n^{1+n} e(Nx^2) dx + Ni^{-N} \sum_{n \in \mathbb{Z}} \int_{n-1/2}^{n+1/2} e(nx^2) dx \\ &= \sqrt{N}(1 + i^{-N}) \int_{-\infty}^{\infty} e(y^2) dy. \end{aligned}$$

The last integral can be checked convergent and we call it  $C$ . Thus,

$$G(1, N) = \sqrt{N}C(1 + i^{-N}).$$

Checking that,  $G(1, 1) = 1$ , we conclude the result.

**1.14. Dirichlet Character.** A *Dirichlet character with modulus  $q$*  is a character

$$\chi : \mathbb{Z}/q\mathbb{Z}^\times \rightarrow \mathbb{C}^\times$$

extended to  $\mathbb{Z}$  by making it  $q$ -periodic and defining  $\chi(a) = 0$  for  $(a, q) > 1$ . Associated to each character  $\chi$ , in addition to its modulus  $q$ , is a natural number  $q'$ , its conductor. The *conductor  $q'$*  is the smallest divisor of  $q$  such that  $\chi$  can be written as  $\chi = \chi' \chi_0$ , where  $\chi_0$  is the trivial Dirichlet character mod  $q$  and  $\chi'$  is a character of modulus  $q'$ . If a character has conductor equal to its modulus then it is called a *primitive Dirichlet character*. Check that, for a primitive Dirichlet character  $\chi$  mod  $q$  one has

$$\frac{1}{q} \sum_{a \pmod q} \chi(ma + b) = \begin{cases} \chi(b), & \text{if } q \mid m \\ 0, & \text{if } q \nmid m. \end{cases}$$

The above is not true for a non-primitive character.

**1.15. Exercise.** Let  $\chi$  be a primitive Dirichlet character mod  $q$  and  $f \in L^1(\mathbb{R})$ . Prove that

$$\sum_{m \in \mathbb{Z}} f(m) \chi(m) = \frac{G(\chi)}{q} \sum_{n \in \mathbb{Z}} \hat{f}(n/q) \bar{\chi}(n),$$

where  $G(\chi)$  is the Gauss sum attached to  $\chi$  defined by

$$G(\chi) := \sum_{a \pmod q} \chi(a) e(a/q).$$

**Hint:** Use the Poisson summation formula.

1.16. **Solution.** First we prove the following. Let  $v \in \mathbb{R}$  and  $u \in \mathbb{R}^+$ . Then using the Poisson summation formula,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} f(um + v) &= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(ux + v) e(-mx) dx \\ &= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(-m(x - v)/u) \frac{dx}{u} \\ &= \frac{1}{u} \sum_{m \in \mathbb{Z}} \hat{f}(m) e(mv/u). \end{aligned}$$

Using the above we get that,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} f(m) \chi(m) &= \sum_{m \in \mathbb{Z}} \sum_{\substack{a \\ a \pmod{q}}} \chi(a) f(mq + a) \\ &= \sum_{\substack{a \\ a \pmod{q}}} \chi(a) \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}(m) e(ma/q) \\ &= \frac{G(\chi)}{q} \sum_{m \in \mathbb{Z}} \hat{f}(m) \bar{\chi}(m). \end{aligned}$$

Here in the last line we have used that for a primitive Dirichlet character  $\chi$ ,

$$\sum_{\substack{a \\ a \pmod{q}}} \chi(a) e(am/q) = \bar{\chi}(m) G(\chi).$$

This can be seen as follows. Let  $(m, q) = 1$ . Then,

$$\bar{\chi}(m) G(\chi) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(am^{-1}) e(a/q) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) e(am/q).$$

If  $(m, q) > 1$  then it follows from the fact that  $\chi(m) = 0$  and

$$\sum_{\substack{a \\ a \pmod{q}}} \chi(a) e(am/q) = \sum_{\substack{y \\ y \pmod{q/(q,m)}}} e(y m/q) \sum_{\substack{x \\ x \pmod{q}}} \chi(xq/d + y) = 0.$$

2. DUE ON 10TH OCTOBER

2.1. **Exercise.** Prove that  $\Gamma(q)$  is a normal subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  and has index in it  $q^3 \prod_{p|q} (1 - \frac{1}{p^2})$ .

2.2. **Solution.** We consider the mod  $q$  reduction map

$$\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}),$$

whose kernel is by definition  $\Gamma(q)$ . Thus  $\Gamma(q)$  is normal. Hence, as the above map is surjective, by the first isomorphism theorem

$$\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(q),$$

and so,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)] = |\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})|.$$

To compute the cardinality we first note that if  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$  then  $(c, d, q) = 1$ . For each such lower row  $(c, d)$  we have exactly  $q$  solutions for the congruence  $ad - bc \equiv 1 \pmod{q}$ . Thus the cardinality is,

$$q|\{(c, d) \pmod{q} \mid (c, d, q) = 1\}| = q \sum_{r|q} \mu(r)(q/r)^2 = q^3 \prod_{p|q} (1 - p^{-2}).$$

2.3. **Exercise.** Recall the subgroups  $\Gamma_0(q)$ ,  $\Gamma_1(q)$  and  $\Gamma_d(q)$  of  $\mathrm{SL}_2(\mathbb{Z})$  from the lectures. Compute indices of the subgroups in  $\mathrm{SL}_2(\mathbb{Z})$ .

2.4. **Solution.** Consider the surjective map

$$\Gamma_1(q) \rightarrow \mathbb{Z}/q\mathbb{Z},$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \pmod{q}.$$

The kernel of this map is by definition  $\Gamma(q)$ . Thus by the first isomorphism theorem,

$$\Gamma_1(q)/\Gamma(q) \cong \mathbb{Z}/q\mathbb{Z}.$$

Hence,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(q)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)][\Gamma_1(q) : \Gamma(q)]^{-1} = q^2 \prod_{p|q} (1 - p^{-2}).$$

Similarly, considering the map

$$\Gamma_0(q) \rightarrow (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod{q},$$

we conclude that

$$\Gamma_0(q)/\Gamma_1(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(q)] = \frac{1}{\phi(q)} q^2 \prod_{p|q} (1 - p^{-2}) = q \prod_{p|q} (1 + p^{-1}).$$

Again similarly, considering the map

$$\Gamma_d(q) \rightarrow (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod{q},$$

we conclude that

$$\Gamma_d(q)/\Gamma(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_d(q)] \frac{1}{\phi(q)} q^3 \prod_{p|q} (1 - p^{-2}) = q^2 \prod_{p|q} (1 + p^{-1}).$$

**2.5. Exercise.** Prove that for any finite abelian group  $G$  one has  $G \cong \hat{G}$ .

**Hint:** First try to show for finite abelian groups  $G_1$  and  $G_2$  that  $\hat{G}_1 \times \hat{G}_2 \cong \widehat{G_1 \times G_2}$ . Then use the structure theory of the finite abelian groups.

**2.6. Solution.** We define a map

$$\hat{G}_1 \times \hat{G}_2 \rightarrow \widehat{G_1 \times G_2} \text{ by } (\chi_1, \chi_2) \mapsto \{\chi : (g_1, g_2) \mapsto \chi_1(g_1)\chi_2(g_2)\}.$$

This map is clearly well-defined homomorphism. To see injectivity if  $\chi$  is the trivial character then

$$\chi_1(g_1) = \chi_2^{-1}(g_2) \forall (g_1, g_2) \in G_1 \times G_2,$$

which implies that  $\chi_i$  are the trivial character. From the lecture we recall that  $|G| = |\hat{G}|$ , which proves the isomorphism. Now from the structure theory of the finite abelian groups we know that every finite abelian group is isomorphic to direct product of  $\mathbb{Z}/n\mathbb{Z}$ . hence it is enough to show that

$$\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \cong \mu_n \cong \mathbb{Z}/n\mathbb{Z},$$

where  $\mu_n$  is the group of  $n$ 'th roots of unity. To See this isomorphism we consider that map

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \rightarrow \mu_n \text{ by } \chi \mapsto \chi(1).$$

This map is clearly a well-defined homomorphism, as  $\chi(1)^n = \chi(n) = \chi(0) = 1$ , i.e.  $\chi(1) \in \mu_n$ . If  $\chi(1) = 1$  then  $\chi(m) = \chi^m(1) = 1$ , which proves the injectivity. Equality of the cardinalities concludes the proof.



2.7. **Exercise.** Recall the the product expansion

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(1) Use the above formula to prove that,

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left[ \frac{1}{z-d} + \frac{1}{z+d} \right] = \pi \cot(\pi z) = \pi i - 2\pi i \sum_{d=0}^{\infty} e(dz).$$

(2) Prove that for even natural number  $k$

$$\zeta(k) = -\frac{(2\pi i)^k}{2k!} B_k,$$

where  $B_k$  are the Bernoulli numbers.

(3) Prove that  $\zeta(s)$  has zeros at negative even integers.

**Hint:** Use the functional equation of  $\zeta(s)$ .

2.8. **Solution.**

(1) We do a logarithmic differentiation of the given expression.

$$\begin{aligned} \pi \cot(\pi z) &= \frac{d}{dz} \log \sin(\pi z) \\ &= \frac{d}{dz} \log(\pi z) + \frac{d}{dz} \sum_{n=1}^{\infty} \log(1 - z^2/n^2) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2}, \end{aligned}$$

hence the first equality. For the second equality we see that,

$$\pi \cot(\pi z) = \pi i \frac{e(z) + 1}{e(z) - 1} = \pi i - 2\pi i \frac{1}{1 - e(z)} = \pi i - 2\pi i \sum_{n=0}^{\infty} e(nz),$$

completing the proof.

(2) Recall that the Bernoulli numbers are defined by the coefficient of the series expansion of  $\frac{x}{e^x - 1}$ , i.e.

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.$$

Consider the generating series of  $\zeta(2k)$

$$1 + 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

For  $|z| < 1$  the above sum is absolutely convergent, so plugging in the definition of  $\zeta(s)$  for  $s > 1$  and changing the order of the summation we get that above sum is

$$1 + 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (z/n)^{2k} = 1 + \sum_{n=1}^{\infty} \frac{2z^2}{n^2 - z^2} = \pi z \cot(\pi z),$$

where the last equality is from (1). But from (2)

$$\pi z \cot(\pi z) = \pi iz - \frac{2\pi iz}{1 - e^{2\pi iz}} = \pi iz - \sum_{k=0}^{\infty} B_k \frac{(2\pi iz)^k}{k!}.$$

Equating two power series we conclude that

$$2\zeta(2k) = -B_{2k} \frac{(2\pi i)^{2k}}{(2k)!},$$

concluding the result.

(3) We recall the functional equation of  $\zeta(s)$

$$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \zeta(1-s)\pi^{(1-s)/2}\Gamma((1-s)/2).$$

We also recall the duplication formula,

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma(s/2)\Gamma((1+s)/2),$$

and

$$\Gamma(1/2 - s/2)\Gamma(1/2 + s/2) = \frac{\pi}{\cos(\pi s/2)}.$$

Combining all of them we get that,

$$\zeta(1-s) = 2(2\pi)^{-s} \cos(\pi s/2)\Gamma(s)\zeta(s).$$

Plugging in  $s = 2n + 1$  for  $n \geq 1$  and checking that  $\cos(n\pi + \pi/2) = 0$  we conclude that

$$\zeta(-2n) = 0.$$

**2.9. Eisenstein Series of weight 2.** In the lecture we have defined Eisenstein series  $E_k$  of weight  $k$  for  $k > 2$ . In this exercise we will define *Eisenstein series*  $E_2$  of weight 2 and will show that it satisfies an “almost modularity” relation.

**2.10. Exercise.** Define the following functions for  $z \in \mathbb{H}$ :

$$\begin{aligned} G_2(z) &:= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2}, \\ G_2^*(z) &:= G_2(z) - \frac{\pi}{2\Im(z)}, \\ G_{2,\epsilon} &:= \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^2} \frac{1}{|mz + n|^{2\epsilon}}, \text{ for } \epsilon > 0. \end{aligned}$$

- (1) Let  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Prove that  $G_{2,\epsilon}$  converges absolutely and locally uniformly. Also show that,

$$G_{2,\epsilon}(\gamma z) = (cz + d)^2 |cz + d|^{2\epsilon} G_{2,\epsilon}(z).$$

- (2) For  $\epsilon > -1/2$  define:

$$I_\epsilon(z) := \int_{\mathbb{R}} \frac{dt}{(z+t)^2 |z+t|^{2\epsilon}} \text{ and } I(\epsilon) := \int_{\mathbb{R}} \frac{dt}{(i+t)^2 (1+t^2)^\epsilon}.$$

Consider

$$G_{2,\epsilon}(z) - \sum_{m=1}^{\infty} I_\epsilon(mz).$$

Use the mean value theorem to prove that it converges absolutely and locally uniformly for  $\epsilon > -1/2$  and the limit as  $\epsilon \rightarrow 0$  is  $G_2(z)$ .

- (3) Show that

$$I_\epsilon(z) = \frac{I(\epsilon)}{\Im(z)^{1+2\epsilon}} \text{ and } I'(0) = -\pi.$$

Use this to show that the limit of  $G_{2,\epsilon}(z)$  as  $\epsilon \rightarrow 0$  is  $G_2^*(z)$ . Hence  $G_2^*$  transforms like a modular form of weight 2.

- (4) Conclude that

$$G_2(\gamma z) = (cz + d)^2 G_2(z) - \pi ic(cz + d).$$

$E_2$  is defined to be, as usual,  $\frac{G_2}{\zeta(2)}$ .

**2.11. Solution.**

- (1) Note that, for  $k > 2$  and  $z \in \mathbb{H}$

$$\sum_{N=1}^{\infty} \sum_{N < |mz+n| \leq N+1} \frac{1}{|mz+n|^k} \leq \sum_{N=1}^{\infty} \frac{\#\{(m,n) \in \mathbb{Z}^2 \mid N \leq |mz+n| \leq N+1\}}{N^k}.$$

It is easy to check that

$$\#\{(m,n) \mid N \leq |mz+n| \leq N+1\} \ll \pi(N+1)^2 - \pi N^2 \ll N.$$

Thus the above sum is, as  $k > 2$

$$\ll \sum_{N=1}^{\infty} N^{1-k} < \infty.$$

Now we see that,

$$G_{2,\epsilon} \leq \sum_{0 \leq |mz+n| \leq 1} |mz+n|^{-2-2\epsilon} + \sum_{1 \leq |mz+n|} |mz+n|^{-2-2\epsilon}.$$

The first sum has finite number of summands and second sum is absolutely and locally uniformly convergent by the previous argument. Thus the sum of  $G_{2,\epsilon}$  are

convergent absolutely and locally uniformly, thus defines a holomorphic function on  $\mathbb{H}$ . To see the transformation law we first note that every  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  induces a bijection from  $\mathbb{Z}^2 \setminus \{(0,0)\}$  to itself by right multiplication. Also one checks that,

$$m\gamma z + n = \frac{(ma + nc)z + (mb + nd)}{cz + d} = \frac{m'z + n'}{cz + d}.$$

Combining these two facts, we conclude that

$$G_{2,\epsilon}(\gamma z) = \sum_{(m',n') \neq (0,0)} \frac{(cz + d)^2 |cz + d|^{2\epsilon}}{(m'z + n') |m'z + n'|^{2\epsilon}} = (cz + d)^2 |cz + d|^{2\epsilon} G_{2,\epsilon}(z).$$

(2) Let

$$f(t) := (mz + t)^2 |mz + t|^{-2\epsilon},$$

with implicit dependence on  $mz$ . Now as we have proved the absolute convergence of the  $\sum f(n)$  we will freely change the order of summations and order of integration and summation, as follows.

$$\begin{aligned} \tilde{G}_{2,\epsilon}(z) &= G_{2,\epsilon}(z) - \sum_{m=0}^{\infty} I_{\epsilon}(mz) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} (f(n) - \int_n^{n+1} f(t) dt) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \int_n^{n+1} (f(n) - f(t)) dt. \end{aligned}$$

By the mean value theorem on  $n \leq t \leq n+1$  we get that

$$|f(n) - f(t)| \leq \sup_{n \leq u \leq n+1} |f'(u)| \ll |mz + n|^{-3-2\epsilon}.$$

Hence, the sum is absolutely convergent for  $\epsilon > -1/2$  and thus  $\lim_{\epsilon \rightarrow 0} \tilde{G}_{2,\epsilon}$  exists and defines a holomorphic function. We calculate,

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \tilde{G}_{2,\epsilon}(z) \\ &= \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m=1}^{\infty} \left[ \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} + \sum_{n \in \mathbb{Z}} \left( \frac{1}{mz + n + 1} - \frac{1}{mz + n} \right) \right] \\ &= \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} \\ &= G_2(z) \end{aligned}$$

(3) Let  $z = x + iy$ . Then changing variable  $t \mapsto yt - x$  we get that,

$$\begin{aligned} I_\epsilon(x + iy) &= \int_{\mathbb{R}} \frac{dt}{(x + t + iy)^2 |x + t + iy|^{2\epsilon}} \\ &= \frac{1}{y^{1+2\epsilon}} \int_{\mathbb{R}} \frac{dt}{(t + i)^2 |t + i|^{2\epsilon}} = \frac{I(\epsilon)}{y^{1+2\epsilon}}. \end{aligned}$$

Differentiating under the integration sign and then integrating by parts we get that,

$$\begin{aligned} I'(0) &= - \int_{\mathbb{R}} \frac{\log(1 + t^2)}{(t + i)^2} dt = \frac{\log(1 + t^2)}{t + i} \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{2t dt}{(t + i)(1 + t^2)} \\ &= - \int_{\mathbb{R}} \frac{1}{(t + i)^2} + \frac{1}{1 + t^2} = - \int_{\mathbb{R}} \frac{dt}{t^2 + 1} = -\pi. \end{aligned}$$

Using the above two results we compute that,

$$\lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} I_\epsilon(mz) = \lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} \frac{I(\epsilon)}{(my)^{1+2\epsilon}} = \lim_{\epsilon \rightarrow 0} \frac{I(\epsilon)\zeta(1 + 2\epsilon)}{\Im(z)^{1+2\epsilon}}.$$

From the exercise 1.6 we know that

$$\zeta(1 + 2\epsilon) = \frac{1}{2\epsilon} + O(1).$$

Using that  $I(0) = 0$  we have that above limit equals to

$$\lim_{\epsilon \rightarrow 0} \frac{I(\epsilon)}{2\epsilon \Im(z)^{1+2\epsilon}} = \frac{I'(0)}{2\Im(z)}.$$

Thus,

$$\lim_{\epsilon \rightarrow 0} G_{2,\epsilon}(Z) = \lim_{\epsilon \rightarrow 0} \left( \tilde{G}_{2,\epsilon}(z) + \sum_{m=1}^{\infty} I_\epsilon(mz) \right) = G_2(z) - \frac{\pi}{2\Im(z)} = G_2^*(z).$$

(4) From part (1) and (3) letting  $\epsilon \rightarrow 0$  we see that  $G_2^*(z)$  transforms as a modular form of weight 2. So,

$$\begin{aligned} G_2(\gamma z) - (cz + d)^2 G_2(z) &= \frac{\pi}{2\Im(\gamma z)} - (cz + d)^2 \frac{\pi}{2\Im(z)} \\ &= \frac{\pi}{2\Im(z)} (|cz + d|^2 - (cz + d)^2) \\ &= \pi ic(cz + d), \end{aligned}$$

concluding the result.

## 3. DUE ON 24TH OCTOBER

3.1. **Exercise.** Prove the Bruhat decomposition: for any subfield  $K \subset \mathbb{C}$

$$\mathrm{SL}_2(K) = N(K)A(K) \sqcup N(K)wN(K)A(K),$$

where the notations are same as in the lectures. Using this prove that the fractional linear transformation  $\mathrm{GL}_2(\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C})$  preserves the lines.

3.2. **Solution.** Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . if  $c = 0$  then  $g$  is upper triangular so lies in  $NA$ . So let us assume that  $c \neq 0$ . So  $b = ad/c$ . Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1/c \end{pmatrix}.$$

This also can be proved in much more geometric way. First check that

$$g.\infty = a/c \implies \mathrm{Stab}_{\mathrm{GL}(2)}(\infty) = NA.$$

We prove that if  $g \notin NA$  then  $g \in NwNA$ . To check this we see that

$$\begin{pmatrix} 1 & -a/c \\ & 1 \end{pmatrix} g.z = g.z - a/c = \frac{az + b}{cz + d} - \frac{a}{c} = \frac{1}{c^2z + cd} = w.c^2z + cd = w \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1/c \end{pmatrix}.z.$$

To check that this decomposition is unique we note that, again, if  $g = b \in NA$  this is obvious. If  $g = nwb = n'wb'$  then

$$g.\infty = n.0 = n'.0 \implies n = n' \implies b = b'.$$

This proves the first part.

For the second part we first recall that a line in  $\mathbb{P}^1(\mathbb{C})$  is of the form  $L \cup \{\infty\}$  where  $L$  is a line or a circle in  $\mathbb{C}$ . As from the previous part and the fact that

$$\mathrm{GL}_2(\mathbb{C}) \cong Z(\mathbb{C})\mathrm{SL}_2(\mathbb{C}),$$

it is enough to prove that  $Z, N, A, w$  preserves the lines. While  $Z, N, A$  transforms in affine way, i.e.

$$z \mapsto az + b, \quad a \in \mathbb{C}^\times, b \in \mathbb{C}$$

it is clear that they preserve lines. Thus it is enough to check that  $w$  preserves a line  $L$ . Now, as we can freely move object in affine way, we may assume that  $L$  is a horizontal line passing through 0, i.e.  $\Im(z) = 0$  or a unit circle centered at origin, i.e.  $|z| = 1$ . In either case the fact that

$$w.z = -\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

proves the claim.

3.3. **Exercise.** Recall the Fourier expansions of the Eisenstein series

$$E_k(z) = 1 + c_k \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where for  $k = 2, 4, \dots, 14$  the  $c_k$  are  $-24, 240, -504, 480, -264, 65520/691, -24$  with  $q := e(z)$  and  $\sigma_s(n) := \sum_{d|n} d^s$ .

- (1) Use dimension formula to show that  $E_8 = E_4^2$ ,  $E_4E_6 = E_{10}$ , and  $E_6E_8 = E_{14}$ . What relations can you get between  $\sigma_n$ 's using the above relations (some of them were obtained during the lectures)?
- (2) Define the *Serre derivative* by

$$D_k := \frac{1}{2\pi i} \frac{d}{dz} - \frac{k}{12} E_2.$$

Show that  $D_k : M_k \rightarrow M_{k+2}$  and  $D_k f \in S_{k+2}$  iff  $f \in S_k$ .

- (3) Calculate  $DE_4$  and  $DE_6$ . Find  $\sigma_5$  in terms of  $\sigma_1$  and  $\sigma_3$  resp. and  $\sigma_7$  in terms of  $\sigma_1$  and  $\sigma_5$ .

3.4. **Solution.**

- (1) Check that from the dimension formula that  $m_8, M_{10}$ , and  $M_{14}$  are one dimensional. Therefore,  $E_8 = cE_4^2$ ,  $E_4E_6 = de_{10}$ , and  $E_6E_8 = eE_{14}$ . But from the Fourier expansions of the Eisenstein series that their first Fourier coefficients are one we conclude that  $c = d = e = 1$ . Now multiplying the Fourier expansions we get that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m),$$

$$-11\sigma_9(n) = 10\sigma_3(n) - 21\sigma_5(n) - 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m),$$

$$-\sigma_{13}(n) = -21\sigma_5(n) + 20\sigma_7(n) - 10080 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_7(n-m).$$

- (2) Let  $f \in M_k$ . As  $E_2, f$ , and  $f'$  are holomorphic so is  $D_k f$ . So it is enough to show that  $D_k f$  transforms as a weight  $k+2$  form to prove that image of  $D_k$  is in  $M_{k+2}$ . We check that for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  with  $j(\gamma, z) = cz + d$ , and recalling from exercise 2.10(4) that

$$E_2(\gamma z) = j(\gamma, z)^2 E_2(z) + \frac{12cj(\gamma, z)}{2\pi i}.$$

we get that

$$\begin{aligned}
D_k f(\gamma z) &= \frac{1}{2\pi i} f'(\gamma z) - \frac{k}{12} E_2(\gamma z) f(\gamma z) \\
&= \frac{1}{2\pi i} j^2(\gamma, z) \frac{df(\gamma z)}{dz} - j^{k+2}(\gamma, z) E_2(z) f(z) - \frac{ck j^{k+1}(\gamma, z)}{2\pi i} f(z) \\
&= \frac{1}{2\pi i} j^2(\gamma, z) \frac{d}{dz} j^k(\gamma, z) f(z) - j^{k+2}(\gamma, z) E_2(z) f(z) - f(z) \frac{j^2(\gamma, z)}{2\pi i} \frac{d}{dz} j^k(\gamma, z) \\
&= \frac{j^{k+2}(\gamma, z)}{2\pi i} f'(z) - j^{k+2}(\gamma, z) E_2(z) f(z) \\
&= j^{k+2}(\gamma, z) D_k f(z).
\end{aligned}$$

Now note that,

$$q = e(z) \implies \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}.$$

Thus if  $f$  has Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

then

$$D_k f = q \frac{df}{dq} - \frac{k}{12} E_2 f = \sum_{n=0}^{\infty} n a_n q^n + \frac{k}{12} E_2 f.$$

Thus it is clear that the zeroth Fourier coefficient is  $-ka_0/12$  and that will be zero if and only if  $a_0 = 0$  which proves the second claim.

- (3) By part (2)  $DE_4 \in M_6$  and  $DE_6 \in M_8$ . From the dimension formulas and the zeroth Fourier coefficients we conclude as in (1) that

$$DE_4 = cE_6, \quad c \in \mathbb{C},$$

with  $c = -1/3$ . Similarly,  $DE_6 = -\frac{1}{2}E_8$ . Now as in (1) comparing the Fourier coefficients we get that

$$21\sigma_5(n) = (30n - 10)\sigma_3(n) + \sigma_1(n) + 240 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_3(n-m),$$

$$20\sigma_7(n) = (42n - 21)\sigma_5(n) + \sigma_1(n) + 504 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_5(n-m).$$



**3.5. Exercise.** Recall that the Delta function from the lecture defined in terms of some Eisenstein series. Here we start with a different definition and show equality afterwards.

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

which has a Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n = q - 24q^2 + 252q^3 + O(q^4) \in \mathbb{Z}[[q]],$$

with  $q = e(z)$  as usual.  $\tau : \mathbb{N} \rightarrow \mathbb{C}$  is called *Ramanujan Tau function*.

- (1) Prove that  $\frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) = E_2(z)$  and conclude that  $\Delta \in S_{12}$ .
- (2) Show that  $\Delta = \frac{E_4^3 - E_6^2}{1728}$ , and derive  $\tau$  in terms of  $\sigma_3$  and  $\sigma_5$ .
- (3) Show that  $E_{12} - E_6^2 = c\Delta$  with  $c = \frac{2^6 3^5 7^2}{691}$  and derive relation between  $\tau$ ,  $\sigma_{11}$  and  $\sigma_5$ .  
Use this to prove the famous congruence by Ramanujan:

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691},$$

for all  $n \geq 1$ .

**3.6. Solution.**

- (1) Recall that  $\frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}$ . Therefore,

$$\begin{aligned} \frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) &= q \frac{d}{dq} \log \left( q \prod_{n=1}^{\infty} (1 - q^n)^{24} \right) \\ &= q \frac{d}{dq} \left[ \log q + 24 \sum_{n=1}^{\infty} 24 \log(1 - q^n) \right] \\ &= q \frac{d}{dq} \left[ \log q - 24 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^{nk}}{k} \right] \\ &= 1 - 24 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} nq^{nk} \\ &= 1 - 24 \sum_{n=1}^{\infty} q^n \left( \sum_{k|n} k \right) = E_2(z). \end{aligned}$$

All interchanges of orders of summations are justified as the series is absolutely convergent as  $|q| < 1$ . Now from the product form it is clear that  $\Delta$  is holomorphic and has zero as zeroth Fourier coefficient. So to prove that  $\Delta \in S_{12}$  it is enough to

show that  $\Delta$  transforms as a weight 12 modular form. To check that keeping the same notations as in the solution 3.4(2) we compute that

$$\begin{aligned}
& \frac{1}{2\pi i} \frac{d}{dz} \log \Delta(\gamma z) \\
&= j(\gamma z)^{-2} \frac{1}{2\pi i} \frac{d}{dz} \log \Delta|_{\gamma z} \\
&= j(\gamma z)^{-2} E_2(\gamma z) \\
&= E_2(z) + \frac{12c}{2\pi i j(\gamma, z)} \\
&= \frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) + \frac{1}{2\pi i} \frac{d}{dz} \log j^{12}(\gamma, z) \\
&= \frac{1}{2\pi i} \frac{d}{dz} \log(j^{12}(\gamma, z)\Delta(z)).
\end{aligned}$$

Thus for each  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  there exists a constant  $0 \neq c(\gamma)$  such that

$$\Delta(\gamma z) = c(\gamma) j^{12}(\gamma, z) \Delta(z).$$

It suffices to show that  $c(\gamma) = 1$  for all  $\gamma$ . It is easy to check that

$$c : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{C}^\times, \quad \gamma \mapsto c(\gamma)$$

a character. Thus it is enough to prove that  $c(T) = 1$  and  $c(S) = 1$  where  $T, S$  are the usual generators of  $\mathrm{SL}_2(\mathbb{Z})$ . But as  $\Delta$  is 1-periodic so  $c(T) = 1$ . Now as  $S \cdot i = i$  and  $\Delta(i) \neq 0$  we see that

$$c(S) = i^{-12} = 1,$$

completing the proof.

- (2) As  $S_{12}$  is one dimensional and  $E_4^3 - E_6^2$  has zero zeroth Fourier coefficient hence,

$$E_4^3 - E_6^2 = d\Delta, \quad d \in \mathbb{C}.$$

$d$  can be calculated to be 1728 from the first Fourier coefficients of  $E_4$  and  $E_6$ . Thus equating Fourier coefficients we conclude that

$$\begin{aligned}
12\tau(n) &= 5\sigma_3(n) + 1200 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) + 96000 \sum_{r=1}^{n-1} \sum_{m=1}^{r-1} \sigma_3(m)\sigma_3(r-m)\sigma_3(n-r) \\
&\quad + 7\sigma_5(n) - 1764 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).
\end{aligned}$$

- (3) Again by dimension formula arguing that  $S_{12}$  is one dimensional and comparing the first Fourier coefficients we conclude that

$$E_{12} - E_6^2 = \frac{2^6 3^5 7^2}{691} \Delta.$$

Comparing the Fourier coefficients we get that

$$2^6 3^5 7^2 \tau(n) = 65520\sigma_{11}(n) + 691.2.504\sigma_5(n) - 691.504^2 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).$$

Dividing by 1008 and reducing mod 691 we conclude that

$$756\tau(n) \equiv 65\tau(n) \equiv 65\sigma_{11}(n) \pmod{691}.$$

As  $(65, 691) = 1$  we conclude the final result.

**3.7. A Riemmanian metric on the upper half plane.** A Riemmanian metric on  $\mathbb{H}$  can be defined as

$$ds^2(z) = \frac{d\Re^2(z) + d\Im^2(z)}{\Im^2(z)},$$

which gives  $\mathbb{H}$  a hyperbolic structure (More details in the upcoming lecture).

**3.8. Exercise.** Let  $z_1, z_2 \in \mathbb{H}$ . We define *geodesic segment between  $z_1$  and  $z_2$*  to be the unique length minimizing curve (which exists) joining  $z_1$  and  $z_2$  under the hyperbolic metric as above. We define the hyperbolic distance between  $z_1$  and  $z_2$  to be

$$d_h(z_1, z_2) := \text{Length of geodesic segment between } z_1 \text{ and } z_2.$$

(1) Prove that

$$ds^2(gz) = ds^2(z), \quad \forall g \in \text{GL}_2^+(\mathbb{R}),$$

that is  $ds^2$  is a  $\text{GL}_2^+(\mathbb{R})$  invariant metric.

(2) Prove that if  $\Re(z_1) = \Re(z_2)$  then the geodesic segment joining them is the vertical line joining  $z_1$  and  $z_2$ .

(3) Prove that for general  $z_1$  and  $z_2$  the geodesic segment joining them is the arc of the unique half-circle centered on  $\mathbb{R}$  containing these two points.

(4) Prove that

$$\cosh(d_h(z_1, z_2)) = 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)}.$$

**3.9. Solution.**

(1) Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Then we check that

$$\frac{d(gz)}{dz} = \frac{\det(g)}{(cz + d)^2}.$$

Also recall that

$$\Im(gz) = \frac{\det(g)\Im(z)}{|cz + d|^2}.$$

Thus

$$ds^2(gz) = \frac{|d(gz)|^2}{\Im(gz)^2} = \frac{|\det(g)|^2}{|cz + d|^4} |dz|^2 \frac{|cz + d|^4}{|\det(g)|^2 \Im(z)^2} = \frac{|dz|^2}{\Im(z)^2} = ds^2(z).$$

- (2) WLOG let  $\Im(z_2) \geq \Im(z_1)$ . Note that, the vertical path joining  $z_1$  and  $z_2$  can be given as

$$\phi(t) = \Re(z_1) + i\Im(z_1) \left( \frac{\Im(z_2)}{\Im(z_1)} \right)^t.$$

It is easy to check that the length of  $\phi$

$$L(\phi) = \log \Im(z_2) - \log \Im(z_1).$$

Let  $\phi'$  be any other curve joining  $z_1$  and  $z_2$ . Then the length of  $\phi_1$

$$L(\phi_1) = \int_0^1 \frac{|\phi_1'(t)|}{\Im(\phi_1(t))} dt \geq \int_0^1 \frac{\Im(\phi_1'(t))}{\Im(\phi_1(t))} dt = \log \Im(z_2) - \log \Im(z_1),$$

which proves the claim.

- (3) First we claim that there exists a  $g \in \mathrm{SL}_2(\mathbb{R})$  such that

$$\Re(gz_1) = \Re(gz_2) = 0.$$

First we assume the claim. Then we see that the length minimizing curve joining  $gz_1$  and  $gz_2$ , them having same real part, is a vertical segment  $\phi$  as in the previous part. As  $\mathrm{SL}_2(\mathbb{R})$  acts by isometry the geodesic joining  $z_1$  and  $z_2$  would be  $g^{-1}\phi$ . From Exercise 3.1 we can conclude that  $\mathrm{SL}_2(\mathbb{R})$  preserves lines in  $\mathbb{P}^1(\mathbb{R}) \cong \mathbb{H} \cup \{\infty\}$ , where lines in  $\mathbb{P}^1(\mathbb{R})$  are vertical lines or half-circles centered in  $\mathbb{R}$ . This concludes the proof assuming the claim.

Now we turn to prove the claim. By transitivity property of  $\mathrm{SL}_2(\mathbb{R})$  action one can find  $g$  such that  $gz_1 = i$ . Now as we know that  $\mathrm{SO}(2)$  fixes  $i$  for any  $k \in \mathrm{SO}(2)$  we have  $gki = z_1$ . So it is enough to find some  $k$  such that  $\Re(kg^{-1}z_2) = 0$ . For any  $z \in \mathbb{H}$  we can always find  $k \in \mathrm{SO}(2)$  such that  $\Re(kz) = 0$ . If  $k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$  and  $z = x + iy$  then to make sure that  $\Re(kz) = 0$  one needs to see whether

$$\tan(2\theta) = -\frac{x}{y^2 + 1 - x^2},$$

which clearly exists.

- (4) By the argument in the part (3) we can find  $g \in \mathrm{SL}_2(\mathbb{R})$  such that  $gz_1$  and  $gz_2$  has zero real parts. Also from part (1) we know that  $g$  acts by isometry thus it is enough to prove the statement for  $z_1$  and  $z_2$  purely imaginary. But in part (2) we have proved that for such  $z_i \in i\mathbb{R}$  one has

$$d_h(z_1, z_2) = |\log \Im(z_1) - \log \Im(z_2)| = |\log(z_1/z_2)|.$$

Thus,

$$\begin{aligned}\cosh(d_h(z_1, z_2)) &= \frac{1}{2} (e^{d_h(z_1, z_2)} + e^{-d_h(z_1, z_2)}) \\ &= \frac{1}{2} \left| \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} \right| 2 = \frac{|z_1^2 + \bar{z}_1^2|}{2|z_1 z_2|} \\ &= 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)},\end{aligned}$$

completing the proof.

## 4. DUE ON 7TH NOVEMBER

4.1. **Exercise.** Prove that all the geodesics of  $\mathbb{H}$  are the perpendicular lines  $\mathbb{P}^1(\mathbb{R})$  at two points.

4.2. **Exercise.** Recall the canonical projection map

$$\pi : \mathbb{H} \rightarrow Y(\Gamma) := \{\Gamma z \mid z \in \mathbb{H}\}.$$

Let  $U_i \subset \mathbb{H}$  be an open set. Prove that

- (1)  $\pi(U_1) \cap \pi(U_2) = \emptyset$  in  $Y(\Gamma)$  iff  $\Gamma(U_1) \cap U_2 = \emptyset$  in  $\mathcal{H}$ .
- (2)  $Y(\Gamma)$  is connected.

4.3. **Exercise.** recall the definition of  $U_{x,Y}$  from the lecture with  $x \in \mathbb{P}^1(\mathbb{Q})$  and  $Y > 0$ . Let  $\Gamma$  be a congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$ ,  $x, y \in \mathbb{P}^1(\mathbb{Q})$ , and  $z \in \mathbb{H}$ .

- (1) Let  $U$  be a neighbourhood of  $z$  with compact closure. Show that the set

$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U \neq \emptyset\}$$

is finite, in fact empty, for  $U$  sufficient small and  $Y$  sufficiently large.

- (2) If  $y \notin \Gamma x$  then show that the set

$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U_{y,Y} \neq \emptyset\}$$

is finite for any  $Y > 0$  and empty if  $Y > 1$ .

- (3) If  $Y > 1$  then show that

$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U_{x,Y} \neq \emptyset\} = \Gamma_x.$$

- (4) Prove that  $X(\Gamma)$  equipped with the quotient topology is a connected, compact, Hausdorff topological space.

4.4. **Exercise.** Show that a set of representatives for  $\mathrm{Cusp}(\Gamma_0(q))$  is given by the fractions

$$\left\{ \frac{u}{v} \mid v \mid q, 0 < u \leq (v, q/v) \right\}.$$

Compute their respective widths.

4.5. **Exercise.** Let  $\mathfrak{a}$  and  $\mathfrak{b}$  are two cusps for a congruence subgroup  $\Gamma$  with scaling matrices  $\sigma_{\mathfrak{a}}$  and  $\sigma_{\mathfrak{b}}$ .

- (1) Prove the following disjoint decomposition of double cosets:

$$\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} = \delta_{\mathfrak{ab}} \cup \bigcup_{c>0} \bigcup_{d \pmod c} B \begin{pmatrix} * & * \\ c & d \end{pmatrix} B,$$

where  $B$  is the set of upper triangular matrices in  $\mathrm{SL}_2(\mathbb{Z})$  and  $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$  such that it belongs to  $\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}$ .

(2) Define

$$C(\mathbf{a}, \mathbf{b}) := \left\{ c > 0 \mid \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathbf{a}}^{-1} \Gamma \sigma_{\mathbf{b}} \right\}.$$

Also define  $c(\mathbf{a}, \mathbf{b})$  to be the smallest element of  $C(\mathbf{a}, \mathbf{b})$ . Prove that for any  $X > 0$

$$\sum_{0 < c \leq X} c^{-1} \left| \left\{ d \pmod{c} \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathbf{a}}^{-1} \Gamma \sigma_{\mathbf{b}} \right\} \right| \leq c(\mathbf{a}, \mathbf{b})^{-1} X.$$

Hence for any  $c \in C(\mathbf{a}, \mathbf{b})$  we have

$$\left| \left\{ d \pmod{c} \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathbf{a}}^{-1} \Gamma \sigma_{\mathbf{b}} \right\} \right| \leq c(\mathbf{a}, \mathbf{b})^{-1} c^2.$$

(3) Let  $z \in \mathbb{H}$  and  $Y > 0$ . Then prove that

$$|\{\gamma \in \Gamma_{\mathbf{a}} \setminus \Gamma \mid \Im \sigma_{\mathbf{a}}^{-1} \gamma z > Y\}| < 1 + \frac{10}{c(\mathbf{a}, \mathbf{a}) Y}.$$