

MODULAR FORMS EXERCISES AND SOLUTIONS

1. DUE ON 26TH SEPTEMBER

1.1. **Exercise.** Let \mathcal{P} be the set of primes. Prove that $\sum_{p \in \mathcal{P}} \frac{1}{p} = +\infty$.

1.2. **Solution.** Let $s > 1$. Then from the Euler product of the Zeta function,

$$\begin{aligned} \log \zeta(s) &= \sum_{p \in \mathcal{P}} -\log(1 - p^{-s}) = \sum_{p \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{1}{k p^{ks}} \\ &\leq \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \sum_{k=2}^{\infty} \frac{1}{p^k} = \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \frac{1}{p(p-1)} \\ &= \sum_{p \in \mathcal{P}} \frac{1}{p^s} + O(1) \end{aligned}$$

As we know that $\lim_{s \rightarrow 1+} \zeta(s) = +\infty$, letting $s \rightarrow 1+$ in the above inequality we conclude that

$$\lim_{s \rightarrow 1+} \sum_{p \in \mathcal{P}} \frac{1}{p^s} = +\infty,$$

hence the result.

1.3. **Summation by Parts.** Let $a : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function, let $0 < y < x$ and let $f : [y, x] \rightarrow \mathbb{C}$ be a function with continuous derivative on $[y, x]$. Then

$$\sum_{y < n \leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where $A(x) = \sum_{n \leq x} a_n$.

1.4. **Exercise.** Prove that for every $\delta > 0$,

$$\pi(x) := |\{p \in \mathcal{P} \mid p \leq x\}|$$

is bigger than $\frac{x}{(\log x)^{1+\delta}}$ for some sufficiently large x .

1.5. **Solution.** Let a_n be the prime indicator function, i.e.

$$a_n := \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not a prime.} \end{cases} .$$

Using summation by parts we note that,

$$\sum_{p \leq x} \frac{1}{p} = \sum_{3/2 < n \leq x} \frac{a_n}{n} = \frac{\pi(x)}{x} + \int_{3/2}^x \frac{\pi(t)}{t^2} dt.$$

If the claim is false i.e. for all sufficiently large x , $\pi(x) \leq x/(\log x)^{1+\delta}$ then from the above,

$$\sum_{p \leq x} \frac{1}{p} \leq \frac{1}{(\log x)^{1+\delta}} + C + \frac{1}{(\log x)^\delta},$$

for some constant C . The RHS of the above tends to C as $x \rightarrow \infty$ contradicting Exercise 1.1, hence the result.

1.6. **Exercise.** Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\}$ is the fractional part of x . Using this show that $\zeta(s)$ has meromorphic continuation to $\Re(s) > 0$ with a simple pole at $s = 1$.

1.7. **Solution.** Let $\Re(s) > 1$. Then using the summation by parts as following.

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n^s} &= \frac{[x]}{x^s} + s \int_1^x \frac{[t]}{t^{s+1}} dt = \frac{1}{x^{s-1}} - \frac{\{x\}}{x^s} + s \int_1^x t^{-s} dt - s \int_1^x \frac{\{t\}}{t^{s+1}} dt \\ &= \frac{s}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}} dt + O(x^{-\Re(s)} + x^{-\Re(s)+1}). \end{aligned}$$

Letting $x \rightarrow \infty$, as $\Re(s) > 1$, we conclude that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$

We now note that the integral right hand side is well defined for $\Re(s) > 0$ and is holomorphic in s . As $\frac{s}{s-1}$ is a meromorphic function with simple pole at $s = 1$ and residue 1, we conclude the meromorphic continuation of $\zeta(s)$ to $\Re(s) > 0$.

1.8. **Exercise.** Prove that the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt$$

has analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple pole at each non-positive integer. Find the residues of the Gamma function at those poles.

Hint: First prove that $\Gamma(s+1) = s\Gamma(s)$.

1.9. **Solution.** By integration by parts we see that

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1} \frac{dt}{t} = \int_0^\infty e^{-t} s t^s \frac{dt}{t} = s\Gamma(s),$$

for $\Re(s) > 0$. Thus $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ extends definition of $\Gamma(s)$ to $\Re(s) > -1$ meromorphically with pole at $s = 0$ as

$$\lim_{s \rightarrow 0^+} \int_0^\infty e^{-t} t^s \frac{dt}{t} = +\infty.$$

The pole is simple, as $\lim_{s \rightarrow 0} s\Gamma(s) = 1$, and with residue 1. Similarly $\Gamma(s)$ can be extended to all $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple poles at $s = -n$, $n \in \mathbb{N}$ with residue,

$$\lim_{s \rightarrow -n} (s+n)\Gamma(s) = \lim_{s \rightarrow -n} \frac{\Gamma(s+n+1)}{(s+n-1)\dots s} = \frac{(-1)^n}{n!}.$$

1.10. **Exercise.** Prove the Poisson summation formula: Let $f \in \mathcal{S}(\mathbb{R})$ be in the Schwartz class. Prove that

$$\sum_{n \in \mathbb{Z}} f(n+u) = \sum_{n \in \mathbb{Z}} \hat{f}(n)e(nu).$$

Note: Putting $u = 0$ we get the usual Poisson summation formula.

1.11. **Solution.** Let

$$F(x) := \sum_{n \in \mathbb{Z}} f(n+x)$$

which is a function on $L^1(\mathbb{R}/\mathbb{Z})$ so has a Fourier expansion of the form

$$F(x) = \sum_{n \in \mathbb{Z}} e(nx)\hat{F}(n).$$

Here

$$\begin{aligned} \hat{F}(n) &= \int_0^1 F(x)e(-nx)dx = \sum_{n \in \mathbb{Z}} \int_0^1 \sum_{m \in \mathbb{Z}} f(m+x)e(-nx)dx \\ &= \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(x)e(-nx)dx = \int_{-\infty}^\infty f(x)e(-nx)dx = \hat{f}(n), \end{aligned}$$

this provides the result.

1.12. **Exercise.** Recall that,

$$G(1, N) := \sum_{n \pmod N} e(n^2/N).$$

Prove that

- (1) For any odd positive integer N , $G(1, N^2) = N$ and $G(1, N^3) = NG(1, N)$.
- (2) For every positive integer N , $G(1, N) = \frac{1+i^{-N}}{1-i} \sqrt{N}$.

1.13. **Solution.** (1) is elementary. We can parametrize the residue class of N^k by

$$\{a_1 N^{k-1} + a_2 N^{k-2} + \cdots + a_k \mid 0 \leq a_i \leq N-1\}.$$

Using this we have,

$$\begin{aligned} G(1, N^2) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(aN+b)^2}{N^2}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \sum_{a=0}^{N-1} e\left(\frac{2ab}{N}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \delta_{b=0} N = N. \end{aligned}$$

Similarly,

$$\begin{aligned} G(1, N^3) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{c=0}^{N-1} e\left(\frac{(aN^2 + bN + c)^2}{N^3}\right) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) \sum_{a=0}^{N-1} e(2ac/N) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) N \delta_{c=0} = NG(1, N). \end{aligned}$$

For the second part we use the Poisson summation formula. First we note the function

$$f(x) := 1_{[0, N]} e(x^2/N)$$

is a function which is continuous on $(0, N)$ and has continuity only from one side at $x = 0, N$. From the Fourier theory we know that the Fourier series of f at $x = 0$ would converge to $\frac{f(0+) + f(0-)}{2} = f(0+)/2$. and similarly, at $x = N$ to $f(N-)/2$. Thus using the (modified) Poisson summation formula and using that $f(0+) = f(N-)$ we get that,

$$\begin{aligned} \sum_{n=0}^N e(N^2/N) &= \frac{f(0+)}{2} + \sum_{n=1}^{N-1} f(n) + \frac{f(N-)}{2} \\ &= \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(nx) dx = \sum_{n \in \mathbb{Z}} \int_0^N e(x^2/N + nx) dx. \end{aligned}$$

Thus,

$$G(1, N) = N \sum_{n \in \mathbb{Z}} \int_0^1 e(Nx^2 + nNx) dx = N \sum_{n \in \mathbb{Z}} e(-Nn^2/4) \int_0^1 e(N(x + n/2)^2)$$

Noting that

$$e(-Nn^2/4) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ i^{-N}, & \text{if } n \text{ is odd.} \end{cases}$$

and dividing the above sum into odd and even parts we get that,

$$\begin{aligned} G(1, N) &= N \sum_{n \in \mathbb{Z}} \int_n^{1+n} e(Nx^2) dx + Ni^{-N} \sum_{n \in \mathbb{Z}} \int_{n-1/2}^{n+1/2} e(nx^2) dx \\ &= \sqrt{N}(1 + i^{-N}) \int_{-\infty}^{\infty} e(y^2) dy. \end{aligned}$$

The last integral can be checked convergent and we call it C . Thus,

$$G(1, N) = \sqrt{N}C(1 + i^{-N}).$$

Checking that, $G(1, 1) = 1$, we conclude the result.

1.14. Dirichlet Character. A *Dirichlet character with modulus q* is a character

$$\chi : \mathbb{Z}/q\mathbb{Z}^\times \rightarrow \mathbb{C}^\times$$

extended to \mathbb{Z} by making it q -periodic and defining $\chi(a) = 0$ for $(a, q) > 1$. Associated to each character χ , in addition to its modulus q , is a natural number q' , its conductor. The *conductor q'* is the smallest divisor of q such that χ can be written as $\chi = \chi' \chi_0$, where χ_0 is the trivial Dirichlet character mod q and χ' is a character of modulus q' . If a character has conductor equal to its modulus then it is called a *primitive Dirichlet character*. Check that, for a primitive Dirichlet character χ mod q one has

$$\frac{1}{q} \sum_{a \pmod q} \chi(ma + b) = \begin{cases} \chi(b), & \text{if } q \mid m \\ 0, & \text{if } q \nmid m. \end{cases}$$

The above is not true for a non-primitive character.

1.15. Exercise. Let χ be a primitive Dirichlet character mod q and $f \in L^1(\mathbb{R})$. Prove that

$$\sum_{m \in \mathbb{Z}} f(m) \chi(m) = \frac{G(\chi)}{q} \sum_{n \in \mathbb{Z}} \hat{f}(n/q) \bar{\chi}(n),$$

where $G(\chi)$ is the Gauss sum attached to χ defined by

$$G(\chi) := \sum_{a \pmod q} \chi(a) e(a/q).$$

Hint: Use the Poisson summation formula.

1.16. **Solution.** First we prove the following. Let $v \in \mathbb{R}$ and $u \in \mathbb{R}^+$. Then using the Poisson summation formula,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} f(um + v) &= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(ux + v) e(-mx) dx \\ &= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(-m(x - v)/u) \frac{dx}{u} \\ &= \frac{1}{u} \sum_{m \in \mathbb{Z}} \hat{f}(m) e(mv/u). \end{aligned}$$

Using the above we get that,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} f(m) \chi(m) &= \sum_{m \in \mathbb{Z}} \sum_{\substack{a \\ a \pmod{q}}} \chi(a) f(mq + a) \\ &= \sum_{\substack{a \\ a \pmod{q}}} \chi(a) \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}(m) e(ma/q) \\ &= \frac{G(\chi)}{q} \sum_{m \in \mathbb{Z}} \hat{f}(m) \bar{\chi}(m). \end{aligned}$$

Here in the last line we have used that for a primitive Dirichlet character χ ,

$$\sum_{\substack{a \\ a \pmod{q}}} \chi(a) e(am/q) = \bar{\chi}(m) G(\chi).$$

This can be seen as follows. Let $(m, q) = 1$. Then,

$$\bar{\chi}(m) G(\chi) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(am^{-1}) e(a/q) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) e(am/q).$$

If $(m, q) > 1$ then it follows from the fact that $\chi(m) = 0$ and

$$\sum_{\substack{a \\ a \pmod{q}}} \chi(a) e(am/q) = \sum_{\substack{y \\ y \pmod{q/(q,m)}}} e(y m/q) \sum_{\substack{x \\ x \pmod{q}}} \chi(xq/d + y) = 0.$$

2. DUE ON 10TH OCTOBER

2.1. **Exercise.** Prove that $\Gamma(q)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and has index in it $q^3 \prod_{p|q} (1 - \frac{1}{p^2})$.

2.2. **Solution.** We consider the mod q reduction map

$$\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}),$$

whose kernel is by definition $\Gamma(q)$. Thus $\Gamma(q)$ is normal. Hence, as the above map is surjective, by the first isomorphism theorem

$$\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(q),$$

and so,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)] = |\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})|.$$

To compute the cardinality we first note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ then $(c, d, q) = 1$. For each such lower row (c, d) we have exactly q solutions for the congruence $ad - bc \equiv 1 \pmod{q}$. Thus the cardinality is,

$$q|\{(c, d) \pmod{q} \mid (c, d, q) = 1\}| = q \sum_{r|q} \mu(r)(q/r)^2 = q^3 \prod_{p|q} (1 - p^{-2}).$$

2.3. **Exercise.** Recall the subgroups $\Gamma_0(q)$, $\Gamma_1(q)$ and $\Gamma_d(q)$ of $\mathrm{SL}_2(\mathbb{Z})$ from the lectures. Compute indices of the subgroups in $\mathrm{SL}_2(\mathbb{Z})$.

2.4. **Solution.** Consider the surjective map

$$\Gamma_1(q) \rightarrow \mathbb{Z}/q\mathbb{Z},$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \pmod{q}.$$

The kernel of this map is by definition $\Gamma(q)$. Thus by the first isomorphism theorem,

$$\Gamma_1(q)/\Gamma(q) \cong \mathbb{Z}/q\mathbb{Z}.$$

Hence,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(q)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)][\Gamma_1(q) : \Gamma(q)]^{-1} = q^2 \prod_{p|q} (1 - p^{-2}).$$

Similarly, considering the map

$$\Gamma_0(q) \rightarrow (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod{q},$$

we conclude that

$$\Gamma_0(q)/\Gamma_1(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(q)] = \frac{1}{\phi(q)} q^2 \prod_{p|q} (1 - p^{-2}) = q \prod_{p|q} (1 + p^{-1}).$$

Again similarly, considering the map

$$\Gamma_d(q) \rightarrow (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod{q},$$

we conclude that

$$\Gamma_d(q)/\Gamma(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_d(q)] \frac{1}{\phi(q)} q^3 \prod_{p|q} (1 - p^{-2}) = q^2 \prod_{p|q} (1 + p^{-1}).$$

2.5. Exercise. Prove that for any finite abelian group G one has $G \cong \hat{G}$.

Hint: First try to show for finite abelian groups G_1 and G_2 that $\hat{G}_1 \times \hat{G}_2 \cong \widehat{G_1 \times G_2}$. Then use the structure theory of the finite abelian groups.

2.6. Solution. We define a map

$$\hat{G}_1 \times \hat{G}_2 \rightarrow \widehat{G_1 \times G_2} \text{ by } (\chi_1, \chi_2) \mapsto \{\chi : (g_1, g_2) \mapsto \chi_1(g_1)\chi_2(g_2)\}.$$

This map is clearly well-defined homomorphism. To see injectivity if χ is the trivial character then

$$\chi_1(g_1) = \chi_2^{-1}(g_2) \forall (g_1, g_2) \in G_1 \times G_2,$$

which implies that χ_i are the trivial character. From the lecture we recall that $|G| = |\hat{G}|$, which proves the isomorphism. Now from the structure theory of the finite abelian groups we know that every finite abelian group is isomorphic to direct product of $\mathbb{Z}/n\mathbb{Z}$. hence it is enough to show that

$$\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \cong \mu_n \cong \mathbb{Z}/n\mathbb{Z},$$

where μ_n is the group of n 'th roots of unity. To See this isomorphism we consider that map

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \rightarrow \mu_n \text{ by } \chi \mapsto \chi(1).$$

This map is clearly a well-defined homomorphism, as $\chi(1)^n = \chi(n) = \chi(0) = 1$, i.e. $\chi(1) \in \mu_n$. If $\chi(1) = 1$ then $\chi(m) = \chi^m(1) = 1$, which proves the injectivity. Equality of the cardinalities concludes the proof.

2.7. **Exercise.** Recall the the product expansion

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(1) Use the above formula to prove that,

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left[\frac{1}{z-d} + \frac{1}{z+d} \right] = \pi \cot(\pi z) = \pi i - 2\pi i \sum_{d=0}^{\infty} e(dz).$$

(2) Prove that for even natural number k

$$\zeta(k) = -\frac{(2\pi i)^k}{2k!} B_k,$$

where B_k are the Bernoulli numbers.

(3) Prove that $\zeta(s)$ has zeros at negative even integers.

Hint: Use the functional equation of $\zeta(s)$.

2.8. **Solution.**

(1) We do a logarithmic differentiation of the given expression.

$$\begin{aligned} \pi \cot(\pi z) &= \frac{d}{dz} \log \sin(\pi z) \\ &= \frac{d}{dz} \log(\pi z) + \frac{d}{dz} \sum_{n=1}^{\infty} \log(1 - z^2/n^2) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2}, \end{aligned}$$

hence the first equality. For the second equality we see that,

$$\pi \cot(\pi z) = \pi i \frac{e(z) + 1}{e(z) - 1} = \pi i - 2\pi i \frac{1}{1 - e(z)} = \pi i - 2\pi i \sum_{n=0}^{\infty} e(nz),$$

completing the proof.

(2) Recall that the Bernoulli numbers are defined by the coefficient of the series expansion of $\frac{x}{e^x - 1}$, i.e.

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.$$

Consider the generating series of $\zeta(2k)$

$$1 + 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

For $|z| < 1$ the above sum is absolutely convergent, so plugging in the definition of $\zeta(s)$ for $s > 1$ and changing the order of the summation we get that above sum is

$$1 + 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (z/n)^{2k} = 1 + \sum_{n=1}^{\infty} \frac{2z^2}{n^2 - z^2} = \pi z \cot(\pi z),$$

where the last equality is from (1). But from (2)

$$\pi z \cot(\pi z) = \pi iz - \frac{2\pi iz}{1 - e^{2\pi iz}} = \pi iz - \sum_{k=0}^{\infty} B_k \frac{(2\pi iz)^k}{k!}.$$

Equating two power series we conclude that

$$2\zeta(2k) = -B_{2k} \frac{(2\pi i)^{2k}}{(2k)!},$$

concluding the result.

(3) We recall the functional equation of $\zeta(s)$

$$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \zeta(1-s)\pi^{(1-s)/2}\Gamma((1-s)/2).$$

We also recall the duplication formula,

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma(s/2)\Gamma((1+s)/2),$$

and

$$\Gamma(1/2 - s/2)\Gamma(1/2 + s/2) = \frac{\pi}{\cos(\pi s/2)}.$$

Combining all of them we get that,

$$\zeta(1-s) = 2(2\pi)^{-s} \cos(\pi s/2)\Gamma(s)\zeta(s).$$

Plugging in $s = 2n + 1$ for $n \geq 1$ and checking that $\cos(n\pi + \pi/2) = 0$ we conclude that

$$\zeta(-2n) = 0.$$

2.9. Eisenstein Series of weight 2. In the lecture we have defined Eisenstein series E_k of weight k for $k > 2$. In this exercise we will define *Eisenstein series E_2 of weight 2* and will show that it satisfies an “almost modularity” relation.

2.10. Exercise. Define the following functions for $z \in \mathbb{H}$:

$$\begin{aligned} G_2(z) &:= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2}, \\ G_2^*(z) &:= G_2(z) - \frac{\pi}{2\Im(z)}, \\ G_{2,\epsilon} &:= \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^2} \frac{1}{|mz + n|^{2\epsilon}}, \text{ for } \epsilon > 0. \end{aligned}$$

- (1) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove that $G_{2,\epsilon}$ converges absolutely and locally uniformly. Also show that,

$$G_{2,\epsilon}(\gamma z) = (cz + d)^2 |cz + d|^{2\epsilon} G_{2,\epsilon}(z).$$

- (2) For $\epsilon > -1/2$ define:

$$I_\epsilon(z) := \int_{\mathbb{R}} \frac{dt}{(z+t)^2 |z+t|^{2\epsilon}} \text{ and } I(\epsilon) := \int_{\mathbb{R}} \frac{dt}{(i+t)^2 (1+t^2)^\epsilon}.$$

Consider

$$G_{2,\epsilon}(z) - \sum_{m=1}^{\infty} I_\epsilon(mz).$$

Use the mean value theorem to prove that it converges absolutely and locally uniformly for $\epsilon > -1/2$ and the limit as $\epsilon \rightarrow 0$ is $G_2(z)$.

- (3) Show that

$$I_\epsilon(z) = \frac{I(\epsilon)}{\Im(z)^{1+2\epsilon}} \text{ and } I'(0) = -\pi.$$

Use this to show that the limit of $G_{2,\epsilon}(z)$ as $\epsilon \rightarrow 0$ is $G_2^*(z)$. Hence G_2^* transforms like a modular form of weight 2.

- (4) Conclude that

$$G_2(\gamma z) = (cz + d)^2 G_2(z) - \pi ic(cz + d).$$

E_2 is defined to be, as usual, $\frac{G_2}{\zeta(2)}$.

2.11. Solution.

- (1) Note that, for $k > 2$ and $z \in \mathbb{H}$

$$\sum_{N=1}^{\infty} \sum_{N < |mz+n| \leq N+1} \frac{1}{|mz+n|^k} \leq \sum_{N=1}^{\infty} \frac{\#\{(m,n) \in \mathbb{Z}^2 \mid N \leq |mz+n| \leq N+1\}}{N^k}.$$

It is easy to check that

$$\#\{(m,n) \mid N \leq |mz+n| \leq N+1\} \ll \pi(N+1)^2 - \pi N^2 \ll N.$$

Thus the above sum is, as $k > 2$

$$\ll \sum_{N=1}^{\infty} N^{1-k} < \infty.$$

Now we see that,

$$G_{2,\epsilon} \leq \sum_{0 \leq |mz+n| \leq 1} |mz+n|^{-2-2\epsilon} + \sum_{1 \leq |mz+n|} |mz+n|^{-2-2\epsilon}.$$

The first sum has finite number of summands and second sum is absolutely and locally uniformly convergent by the previous argument. Thus the sum of $G_{2,\epsilon}$ are

convergent absolutely and locally uniformly, thus defines a holomorphic function on \mathbb{H} . To see the transformation law we first note that every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ induces a bijection from $\mathbb{Z}^2 \setminus \{(0,0)\}$ to itself by right multiplication. Also one checks that,

$$m\gamma z + n = \frac{(ma + nc)z + (mb + nd)}{cz + d} = \frac{m'z + n'}{cz + d}.$$

Combining these two facts, we conclude that

$$G_{2,\epsilon}(\gamma z) = \sum_{(m',n') \neq (0,0)} \frac{(cz + d)^2 |cz + d|^{2\epsilon}}{(m'z + n') |m'z + n'|^{2\epsilon}} = (cz + d)^2 |cz + d|^{2\epsilon} G_{2,\epsilon}(z).$$

(2) Let

$$f(t) := (mz + t)^2 |mz + t|^{-2\epsilon},$$

with implicit dependence on mz . Now as we have proved the absolute convergence of the $\sum f(n)$ we will freely change the order of summations and order of integration and summation, as follows.

$$\begin{aligned} \tilde{G}_{2,\epsilon}(z) &= G_{2,\epsilon}(z) - \sum_{m=0}^{\infty} I_{\epsilon}(mz) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} (f(n) - \int_n^{n+1} f(t) dt) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \int_n^{n+1} (f(n) - f(t)) dt. \end{aligned}$$

By the mean value theorem on $n \leq t \leq n+1$ we get that

$$|f(n) - f(t)| \leq \sup_{n \leq u \leq n+1} |f'(u)| \ll |mz + n|^{-3-2\epsilon}.$$

Hence, the sum is absolutely convergent for $\epsilon > -1/2$ and thus $\lim_{\epsilon \rightarrow 0} \tilde{G}_{2,\epsilon}$ exists and defines a holomorphic function. We calculate,

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \tilde{G}_{2,\epsilon}(z) \\ &= \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m=1}^{\infty} \left[\sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} + \sum_{n \in \mathbb{Z}} \left(\frac{1}{mz + n + 1} - \frac{1}{mz + n} \right) \right] \\ &= \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} \\ &= G_2(z) \end{aligned}$$

(3) Let $z = x + iy$. Then changing variable $t \mapsto yt - x$ we get that,

$$\begin{aligned} I_\epsilon(x + iy) &= \int_{\mathbb{R}} \frac{dt}{(x + t + iy)^2 |x + t + iy|^{2\epsilon}} \\ &= \frac{1}{y^{1+2\epsilon}} \int_{\mathbb{R}} \frac{dt}{(t + i)^2 |t + i|^{2\epsilon}} = \frac{I(\epsilon)}{y^{1+2\epsilon}}. \end{aligned}$$

Differentiating under the integration sign and then integrating by parts we get that,

$$\begin{aligned} I'(0) &= - \int_{\mathbb{R}} \frac{\log(1 + t^2)}{(t + i)^2} dt = \frac{\log(1 + t^2)}{t + i} \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{2tdt}{(t + i)(1 + t^2)} \\ &= - \int_{\mathbb{R}} \frac{1}{(t + i)^2} + \frac{1}{1 + t^2} = - \int_{\mathbb{R}} \frac{dt}{t^2 + 1} = -\pi. \end{aligned}$$

Using the above two results we compute that,

$$\lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} I_\epsilon(mz) = \lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} \frac{I(\epsilon)}{(my)^{1+2\epsilon}} = \lim_{\epsilon \rightarrow 0} \frac{I(\epsilon)\zeta(1 + 2\epsilon)}{\Im(z)^{1+2\epsilon}}.$$

From the exercise 1.6 we know that

$$\zeta(1 + 2\epsilon) = \frac{1}{2\epsilon} + O(1).$$

Using that $I(0) = 0$ we have that above limit equals to

$$\lim_{\epsilon \rightarrow 0} \frac{I(\epsilon)}{2\epsilon\Im(z)^{1+2\epsilon}} = \frac{I'(0)}{2\Im(z)}.$$

Thus,

$$\lim_{\epsilon \rightarrow 0} G_{2,\epsilon}(Z) = \lim_{\epsilon \rightarrow 0} \left(\tilde{G}_{2,\epsilon}(z) + \sum_{m=1}^{\infty} I_\epsilon(mz) \right) = G_2(z) - \frac{\pi}{2\Im(z)} = G_2^*(z).$$

(4) From part (1) and (3) letting $\epsilon \rightarrow 0$ we see that $G_2^*(z)$ transforms as a modular form of weight 2. So,

$$\begin{aligned} G_2(\gamma z) - (cz + d)^2 G_2(z) &= \frac{\pi}{2\Im(\gamma z)} - (cz + d)^2 \frac{\pi}{2\Im(z)} \\ &= \frac{\pi}{2\Im(z)} (|cz + d|^2 - (cz + d)^2) \\ &= \pi ic(cz + d), \end{aligned}$$

concluding the result.

3. DUE ON 24TH OCTOBER

3.1. **Exercise.** Prove the Bruhat decomposition: for any subfield $K \subset \mathbb{C}$

$$\mathrm{SL}_2(K) = N(K)A(K) \sqcup N(K)wN(K)A(K),$$

where the notations are same as in the lectures. Using this prove that the fractional linear transformation $\mathrm{GL}_2(\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C})$ preserves the lines.

3.2. **Solution.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. if $c = 0$ then g is upper triangular so lies in NA . So let us assume that $c \neq 0$. So $b = ad/c$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1/c \end{pmatrix}.$$

This also can be proved in much more geometric way. First check that

$$g.\infty = a/c \implies \mathrm{Stab}_{\mathrm{GL}(2)}(\infty) = NA.$$

We prove that if $g \notin NA$ then $g \in NwNA$. To check this we see that

$$\begin{pmatrix} 1 & -a/c \\ & 1 \end{pmatrix} g.z = g.z - a/c = \frac{az + b}{cz + d} - \frac{a}{c} = \frac{1}{c^2z + cd} = w.c^2z + cd = w \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1/c \end{pmatrix}.z.$$

To check that this decomposition is unique we note that, again, if $g = b \in NA$ this is obvious. If $g = nwb = n'wb'$ then

$$g.\infty = n.0 = n'.0 \implies n = n' \implies b = b'.$$

This proves the first part.

For the second part we first recall that a line in $\mathbb{P}^1(\mathbb{C})$ is of the form $L \cup \{\infty\}$ where L is a line or a circle in \mathbb{C} . As from the previous part and the fact that

$$\mathrm{GL}_2(\mathbb{C}) \cong Z(\mathbb{C})\mathrm{SL}_2(\mathbb{C}),$$

it is enough to prove that Z, N, A, w preserves the lines. While Z, N, A transforms in affine way, i.e.

$$z \mapsto az + b, \quad a \in \mathbb{C}^\times, b \in \mathbb{C}$$

it is clear that they preserve lines. Thus it is enough to check that w preserves a line L . Now, as we can freely move object in affine way, we may assume that L is a horizontal line passing through 0, i.e. $\Im(z) = 0$ or a unit circle centered at origin, i.e. $|z|=1$. In either case the fact that

$$w.z = -\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

proves the claim.

3.3. **Exercise.** Recall the Fourier expansions of the Eisenstein series

$$E_k(z) = 1 + c_k \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where for $k = 2, 4, \dots, 14$ the c_k are $-24, 240, -504, 480, -264, 65520/691, -24$ with $q := e(z)$ and $\sigma_s(n) := \sum_{d|n} d^s$.

- (1) Use dimension formula to show that $E_8 = E_4^2$, $E_4E_6 = E_{10}$, and $E_6E_8 = E_{14}$. What relations can you get between σ_n 's using the above relations (some of them were obtained during the lectures)?
- (2) Define the *Serre derivative* by

$$D_k := \frac{1}{2\pi i} \frac{d}{dz} - \frac{k}{12} E_2.$$

Show that $D_k : M_k \rightarrow M_{k+2}$ and $D_k f \in S_{k+2}$ iff $f \in S_k$.

- (3) Calculate DE_4 and DE_6 . Find σ_5 in terms of σ_1 and σ_3 resp. and σ_7 in terms of σ_1 and σ_5 .

3.4. **Solution.**

- (1) Check that from the dimension formula that m_8, M_{10} , and M_{14} are one dimensional. Therefore, $E_8 = cE_4^2$, $E_4E_6 = de_{10}$, and $E_6E_8 = eE_{14}$. But from the Fourier expansions of the Eisenstein series that their first Fourier coefficients are one we conclude that $c = d = e = 1$. Now multiplying the Fourier expansions we get that

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m),$$

$$-11\sigma_9(n) = 10\sigma_3(n) - 21\sigma_5(n) - 5040 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m),$$

$$-\sigma_{13}(n) = -21\sigma_5(n) + 20\sigma_7(n) - 10080 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_7(n-m).$$

- (2) Let $f \in M_k$. As E_2, f , and f' are holomorphic so is $D_k f$. So it is enough to show that $D_k f$ transforms as a weight $k+2$ form to prove that image of D_k is in M_{k+2} . We check that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $j(\gamma, z) = cz + d$, and recalling from exercise 2.10(4) that

$$E_2(\gamma z) = j(\gamma, z)^2 E_2(z) + \frac{12cj(\gamma, z)}{2\pi i}.$$

we get that

$$\begin{aligned}
D_k f(\gamma z) &= \frac{1}{2\pi i} f'(\gamma z) - \frac{k}{12} E_2(\gamma z) f(\gamma z) \\
&= \frac{1}{2\pi i} j^2(\gamma, z) \frac{df(\gamma z)}{dz} - j^{k+2}(\gamma, z) E_2(z) f(z) - \frac{ck j^{k+1}(\gamma, z)}{2\pi i} f(z) \\
&= \frac{1}{2\pi i} j^2(\gamma, z) \frac{d}{dz} j^k(\gamma, z) f(z) - j^{k+2}(\gamma, z) E_2(z) f(z) - f(z) \frac{j^2(\gamma, z)}{2\pi i} \frac{d}{dz} j^k(\gamma, z) \\
&= \frac{j^{k+2}(\gamma, z)}{2\pi i} f'(z) - j^{k+2}(\gamma, z) E_2(z) f(z) \\
&= j^{k+2}(\gamma, z) D_k f(z).
\end{aligned}$$

Now note that,

$$q = e(z) \implies \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}.$$

Thus if f has Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

then

$$D_k f = q \frac{df}{dq} - \frac{k}{12} E_2 f = \sum_{n=0}^{\infty} n a_n q^n + \frac{k}{12} E_2 f.$$

Thus it is clear that the zeroth Fourier coefficient is $-ka_0/12$ and that will be zero if and only if $a_0 = 0$ which proves the second claim.

- (3) By part (2) $DE_4 \in M_6$ and $DE_6 \in M_8$. From the dimension formulas and the zeroth Fourier coefficients we conclude as in (1) that

$$DE_4 = cE_6, \quad c \in \mathbb{C},$$

with $c = -1/3$. Similarly, $DE_6 = -\frac{1}{2}E_8$. Now as in (1) comparing the Fourier coefficients we get that

$$21\sigma_5(n) = (30n - 10)\sigma_3(n) + \sigma_1(n) + 240 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_3(n-m),$$

$$20\sigma_7(n) = (42n - 21)\sigma_5(n) + \sigma_1(n) + 504 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_5(n-m).$$

3.5. Exercise. Recall that the Delta function from the lecture defined in terms of some Eisenstein series. Here we start with a different definition and show equality afterwards.

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

which has a Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n = q - 24q^2 + 252q^3 + O(q^4) \in \mathbb{Z}[[q]],$$

with $q = e(z)$ as usual. $\tau : \mathbb{N} \rightarrow \mathbb{C}$ is called *Ramanujan Tau function*.

- (1) Prove that $\frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) = E_2(z)$ and conclude that $\Delta \in S_{12}$.
- (2) Show that $\Delta = \frac{E_4^3 - E_6^2}{1728}$, and derive τ in terms of σ_3 and σ_5 .
- (3) Show that $E_{12} - E_6^2 = c\Delta$ with $c = \frac{2^6 3^5 7^2}{691}$ and derive relation between τ , σ_{11} and σ_5 .
Use this to prove the famous congruence by Ramanujan:

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691},$$

for all $n \geq 1$.

3.6. Solution.

- (1) Recall that $\frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}$. Therefore,

$$\begin{aligned} \frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) &= q \frac{d}{dq} \log \left(q \prod_{n=1}^{\infty} (1 - q^n)^{24} \right) \\ &= q \frac{d}{dq} \left[\log q + 24 \sum_{n=1}^{\infty} 24 \log(1 - q^n) \right] \\ &= q \frac{d}{dq} \left[\log q - 24 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{q^{nk}}{k} \right] \\ &= 1 - 24 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} nq^{nk} \\ &= 1 - 24 \sum_{n=1}^{\infty} q^n \left(\sum_{k|n} k \right) = E_2(z). \end{aligned}$$

All interchanges of orders of summations are justified as the series is absolutely convergent as $|q| < 1$. Now from the product form it is clear that Δ is holomorphic and has zero as zeroth Fourier coefficient. So to prove that $\Delta \in S_{12}$ it is enough to

show that Δ transforms as a weight 12 modular form. To check that keeping the same notations as in the solution 3.4(2) we compute that

$$\begin{aligned}
& \frac{1}{2\pi i} \frac{d}{dz} \log \Delta(\gamma z) \\
&= j(\gamma z)^{-2} \frac{1}{2\pi i} \frac{d}{dz} \log \Delta|_{\gamma z} \\
&= j(\gamma z)^{-2} E_2(\gamma z) \\
&= E_2(z) + \frac{12c}{2\pi i j(\gamma, z)} \\
&= \frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) + \frac{1}{2\pi i} \frac{d}{dz} \log j^{12}(\gamma, z) \\
&= \frac{1}{2\pi i} \frac{d}{dz} \log(j^{12}(\gamma, z)\Delta(z)).
\end{aligned}$$

Thus for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exists a constant $0 \neq c(\gamma)$ such that

$$\Delta(\gamma z) = c(\gamma) j^{12}(\gamma, z) \Delta(z).$$

It suffices to show that $c(\gamma) = 1$ for all γ . It is easy to check that

$$c : \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathbb{C}^\times, \quad \gamma \mapsto c(\gamma)$$

a character. Thus it is enough to prove that $c(T) = 1$ and $c(S) = 1$ where T, S are the usual generators of $\mathrm{SL}_2(\mathbb{Z})$. But as Δ is 1-periodic so $c(T) = 1$. Now as $S \cdot i = i$ and $\Delta(i) \neq 0$ we see that

$$c(S) = i^{-12} = 1,$$

completing the proof.

- (2) As S_{12} is one dimensional and $E_4^3 - E_6^2$ has zero zeroth Fourier coefficient hence,

$$E_4^3 - E_6^2 = d\Delta, \quad d \in \mathbb{C}.$$

d can be calculated to be 1728 from the first Fourier coefficients of E_4 and E_6 . Thus equating Fourier coefficients we conclude that

$$\begin{aligned}
12\tau(n) &= 5\sigma_3(n) + 1200 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) + 96000 \sum_{r=1}^{n-1} \sum_{m=1}^{r-1} \sigma_3(m)\sigma_3(r-m)\sigma_3(n-r) \\
&\quad + 7\sigma_5(n) - 1764 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).
\end{aligned}$$

- (3) Again by dimension formula arguing that S_{12} is one dimensional and comparing the first Fourier coefficients we conclude that

$$E_{12} - E_6^2 = \frac{2^6 3^5 7^2}{691} \Delta.$$

Comparing the Fourier coefficients we get that

$$2^6 3^5 7^2 \tau(n) = 65520\sigma_{11}(n) + 691.2.504\sigma_5(n) - 691.504^2 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).$$

Dividing by 1008 and reducing mod 691 we conclude that

$$756\tau(n) \equiv 65\tau(n) \equiv 65\sigma_{11}(n) \pmod{691}.$$

As $(65, 691) = 1$ we conclude the final result.

3.7. A Riemmanian metric on the upper half plane. A Riemmanian metric on \mathbb{H} can be defined as

$$ds^2(z) = \frac{d\Re^2(z) + d\Im^2(z)}{\Im^2(z)},$$

which gives \mathbb{H} a hyperbolic structure (More details in the upcoming lecture).

3.8. Exercise. Let $z_1, z_2 \in \mathbb{H}$. We define *geodesic segment between z_1 and z_2* to be the unique length minimizing curve (which exists) joining z_1 and z_2 under the hyperbolic metric as above. We define the hyperbolic distance between z_1 and z_2 to be

$$d_h(z_1, z_2) := \text{Length of geodesic segment between } z_1 \text{ and } z_2.$$

(1) Prove that

$$ds^2(gz) = ds^2(z), \quad \forall g \in \text{GL}_2^+(\mathbb{R}),$$

that is ds^2 is a $\text{GL}_2^+(\mathbb{R})$ invariant metric.

(2) Prove that if $\Re(z_1) = \Re(z_2)$ then the geodesic segment joining them is the vertical line joining z_1 and z_2 .

(3) Prove that for general z_1 and z_2 the geodesic segment joining them is the arc of the unique half-circle centered on \mathbb{R} containing these two points.

(4) Prove that

$$\cosh(d_h(z_1, z_2)) = 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)}.$$

3.9. Solution.

(1) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we check that

$$\frac{d(gz)}{dz} = \frac{\det(g)}{(cz + d)^2}.$$

Also recall that

$$\Im(gz) = \frac{\det(g)\Im(z)}{|cz + d|^2}.$$

Thus

$$ds^2(gz) = \frac{|d(gz)|^2}{\Im(gz)^2} = \frac{|\det(g)|^2}{|cz + d|^4} |dz|^2 \frac{|cz + d|^4}{|\det(g)|^2 \Im(z)^2} = \frac{|dz|^2}{\Im(z)^2} = ds^2(z).$$

- (2) WLOG let $\Im(z_2) \geq \Im(z_1)$. Note that, the vertical path joining z_1 and z_2 can be given as

$$\phi(t) = \Re(z_1) + i\Im(z_1) \left(\frac{\Im(z_2)}{\Im(z_1)} \right)^t.$$

It is easy to check that the length of ϕ

$$L(\phi) = \log \Im(z_2) - \log \Im(z_1).$$

Let ϕ' be any other curve joining z_1 and z_2 . Then the length of ϕ_1

$$L(\phi_1) = \int_0^1 \frac{|\phi_1'(t)|}{\Im(\phi_1(t))} dt \geq \int_0^1 \frac{\Im(\phi_1'(t))}{\Im(\phi_1(t))} dt = \log \Im(z_2) - \log \Im(z_1),$$

which proves the claim.

- (3) First we claim that there exists a $g \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\Re(gz_1) = \Re(gz_2) = 0.$$

First we assume the claim. Then we see that the length minimizing curve joining gz_1 and gz_2 , them having same real part, is a vertical segment ϕ as in the previous part. As $\mathrm{SL}_2(\mathbb{R})$ acts by isometry the geodesic joining z_1 and z_2 would be $g^{-1}\phi$. From Exercise 3.1 we can conclude that $\mathrm{SL}_2(\mathbb{R})$ preserves lines in $\mathbb{P}^1(\mathbb{R}) \cong \mathbb{H} \cup \{\infty\}$, where lines in $\mathbb{P}^1(\mathbb{R})$ are vertical lines or half-circles centered in \mathbb{R} . This concludes the proof assuming the claim.

Now we turn to prove the claim. By transitivity property of $\mathrm{SL}_2(\mathbb{R})$ action one can find g such that $gz_1 = i$. Now as we know that $\mathrm{SO}(2)$ fixes i for any $k \in \mathrm{SO}(2)$ we have $gki = z_1$. So it is enough to find some k such that $\Re(kg^{-1}z_2) = 0$. For any $z \in \mathbb{H}$ we can always find $k \in \mathrm{SO}(2)$ such that $\Re(kz) = 0$. If $k = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ and $z = x + iy$ then to make sure that $\Re(kz) = 0$ one needs to see whether

$$\tan(2\theta) = -\frac{x}{y^2 + 1 - x^2},$$

which clearly exists.

- (4) By the argument in the part (3) we can find $g \in \mathrm{SL}_2(\mathbb{R})$ such that gz_1 and gz_2 has zero real parts. Also from part (1) we know that g acts by isometry thus it is enough to prove the statement for z_1 and z_2 purely imaginary. But in part (2) we have proved that for such $z_i \in i\mathbb{R}$ one has

$$d_h(z_1, z_2) = |\log \Im(z_1) - \log \Im(z_2)| = |\log(z_1/z_2)|.$$

Thus,

$$\begin{aligned}\cosh(d_h(z_1, z_2)) &= \frac{1}{2} (e^{d_h(z_1, z_2)} + e^{-d_h(z_1, z_2)}) \\ &= \frac{1}{2} \left| \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} \right| 2 = \frac{|z_1^2 + z_2^2|}{2|z_1 z_2|} \\ &= 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)},\end{aligned}$$

completing the proof.

4. DUE ON 7TH NOVEMBER

4.1. **Exercise.** Prove that all the geodesics of \mathbb{H} are the perpendicular lines $\mathbb{P}^1(\mathbb{R})$ at two points.

4.2. **Solution.** From the Exercise 3.8(2), 3.8(3) we know the the geodesic joining two points z_1 and z_2 in \mathbb{H} is the arc of the unique half-circle centered on \mathbb{R} containing these two points if they have different real parts and the vertical lines joining them if they have same real parts. So we need to check that both the semicircles centered in \mathbb{R} and the vertical lines are perpendicular to $\mathbb{P}^1(\mathbb{R})$ at two points. As $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, semicircles are clearly perpendicular to \mathbb{R} at two points in \mathbb{R} , where as, vertical lines are perpendicular at a point in \mathbb{R} and ∞ . This proves the claim.

4.3. **Exercise.** Recall the canonical projection map

$$\pi : \mathbb{H} \rightarrow Y(\Gamma) := \{\Gamma z \mid z \in \mathbb{H}\}.$$

Let $U_i \subset \mathbb{H}$ be an open set. Prove that

- (1) $\pi(U_1) \cap \pi(U_2) = \emptyset$ in $Y(\Gamma)$ iff $\Gamma(U_1) \cap U_2 = \emptyset$ in \mathbb{H} .
- (2) $Y(\Gamma)$ is connected.

4.4. **Solution.**

- (1) We will show the contrapositive, that

$$\pi(U_1) \cap \pi(U_2) \neq \emptyset \iff \Gamma(U_1) \cap U_2 \neq \emptyset.$$

To see this let for some $u_i \in U_i$ for $i = 1, 2$

$$\Gamma u_1 = \Gamma u_2.$$

This implies that for $\gamma_1 \in \Gamma$ there exists $\gamma_2 \in \Gamma$ such that

$$\gamma_1 u_1 = \gamma_2 u_2 \implies \gamma_2^{-1} \gamma_1 u_1 = u_2.$$

But the above implies that $u_2 \in \Gamma u_1$, in other words,

$$u_2 \in \Gamma(U_1) \cap U_2.$$

The opposite implication is trivial. If there exists $u_i \in U_i$ for $i = 1, 2$ such that $u_2 \in \Gamma u_1$ then $\Gamma u_1 = \Gamma u_2$. Hence $\pi(U_1) \cap \pi(U_2) \neq \emptyset$.

- (2) As π is a projection, hence a continuous surjection, and \mathbb{H} is connected so $\text{Im}(\pi) = Y(\Gamma)$ is also connected.

4.5. **Exercise.** Recall the definition of $U_{x,Y}$ from the lecture with $x \in \mathbb{P}^1(\mathbb{Q})$ and $Y > 0$. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, $x, y \in \mathbb{P}^1(\mathbb{Q})$, and $z \in \mathbb{H}$.

(1) Let U be a neighbourhood of z with compact closure. Show that the set

$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U \neq \emptyset\}$$

is finite, in fact empty, for U sufficient small and Y sufficiently large.

(2) If $y \notin \Gamma x$ then show that the set

$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U_{y,Y} \neq \emptyset\}$$

is finite for any $Y > 0$ and empty if $Y > 1$.

(3) If $Y > 1$ then show that

$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U_{x,Y} \neq \emptyset\} = \Gamma_x.$$

(4) Prove that $X(\Gamma)$ equipped with the quotient topology is a connected, compact, Hausdorff topological space.

4.6. **Solution.**

(1) Note that

$$\gamma U_{x,Y} = \gamma \sigma_x(H_Y \cup \{\infty\}) = \gamma \sigma_x H_Y \cup \gamma \{x\}.$$

As $x \notin \mathbb{H}$ and U has compact closure hence

$$u \in \gamma U_{x,Y} \cap U \implies u \in \gamma \sigma_x H_Y.$$

Now as σ_x is determined up to a translation on the right and H_Y is translation invariant we may think that $x = \infty$ and count $\gamma \in \Gamma$ so that $\gamma u \in H_Y$ with $u \in U$. In the usual notation of γ this implies that

$$\frac{\Im(u)}{|cu + d|^2} > Y \implies \min \left\{ \frac{1}{c^2 \Im(u)}, \frac{\Im(u)}{(c\Re(u) + d)^2} \right\} > Y.$$

As U has compact closure both c and d has finitely many choices, following a similar argument as in the lecture. Thus there are only finitely many (c, d) such that γ with bottom row (c, d) has $\gamma U_{x,Y} \cap U \neq \emptyset$. This in turn, equivalently, implies that there are finitely many $\gamma \in \Gamma_x \setminus \Gamma$ such that the same happens. In fact, if

$$Y > \sup_{u \in U} \{\Im(u), \Im(u)^{-1}\},$$

then from the above inequalities we conclude that $c = 0 = d$, thus no possible choice for γ .

(2) By conjugating we may assume that $y = \infty$. So, as $U_{x,Y} = \sigma_x H_Y \cup \{x\}$ and $\infty \notin \Gamma x$, we have that

$$u \in \gamma U_{x,Y} \cap U_{\infty,Y} \implies \gamma^{-1} \sigma_x^{-1} u, u \in H_Y.$$

We count $\gamma' := \gamma^{-1} \sigma_x^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Again proceeding as previous we see that c has finitely many choice and thus finitely many choices for γ . This proves the claim.

- (3) As we know that $\sigma_x^{-1}x = \infty$ and $\sigma_x^{-1}\Gamma_x\sigma_x = B$ we may conjugate the claimed equation by σ_x and assume that $x = \infty$. Therefore, if $u \in \gamma^{-1}U_{x,Y} \cap U_{x,Y}$ we have that $\mathfrak{S}(\gamma u) > Y$ and $\mathfrak{S}(u) > Y$. This implies that

$$Y > \mathfrak{S}(\gamma u) = \frac{\mathfrak{S}(u)}{|cu + d|^2} \geq \frac{1}{c^2\mathfrak{S}(u)} < c^{-2}Y^{-1}.$$

As $Y > 1$ this implies that $c = 0$ and thus $\gamma \in B$. Other inclusion is trivial to show.

- (4) Compactness and connectedness of $X(\Gamma)$ follow from compactification and connectedness of compactification of connected space respectively. Hausdorff property follows from part (1) and (2).

4.7. **Exercise.** Show that a set of representatives for $\text{Cusp}(\Gamma_0(q))$ is given by the fractions

$$\left\{ \frac{u}{v} \mid v \mid q, 0 < u \leq (v, q/v) \right\}.$$

Compute their respective widths.

4.8. **Solution.** All cusps are equivalent to some rational numbers as $\Gamma_0(q) \subset \text{SL}_2(\mathbb{Z})$. First we find a set of representatives of $\Gamma_0(q) \backslash \text{SL}_2(\mathbb{Z})$. We claim that they are given by

$$\begin{pmatrix} a & * \\ c & * \end{pmatrix}, \quad \text{with } c \mid q, (a, c) = 1 \text{ and } a \pmod{q/c}.$$

To see this we check that

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ c'a + d'c & c'b + d'd \end{pmatrix}.$$

Here $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(q)$. So (a, q) is invariant by multiplication of $\Gamma_0(q)$ in the left. In fact, we can choose c', d' such that

$$v := c'b + d'd = (d, q) \implies v \mid q.$$

Solutions of $c'b + d'd = v$ form an one parameter family $(c' + dt, d' - bt)$ where t ranges over $\text{mod } q/v$, to ensure $c' \equiv 0 \pmod{q}$. These solutions translate the bottom left $u := c'a + d'cb$ by t , which ensures any choice of u modulo q/v . So a set of representatives can be chosen as

$$\tau = \begin{pmatrix} * & * \\ a & c \end{pmatrix}, \quad \text{with } c \mid q, (a, c) = 1 \text{ and } a \pmod{q/c}.$$

Now transforming

$$\tau \mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \tau^{-1}$$

we conclude the claim.

Now the cusps of $\Gamma_0(q)$ are $\Gamma_0(q) \backslash \text{SL}_2(\mathbb{Z}) \cdot \infty$. They are of the form

$$\{a/c \mid \text{with } c \mid q, (a, c) = 1 \text{ and } a \pmod{(c, q/c)}\}.$$

To see that they are non-equivalent, let

$$\frac{a'}{c'} = \gamma \cdot \frac{a}{c},$$

for some $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_0(q)$. So $c' = c_1 a + d_1 c$. This implies that $c \mid c'$ and as $(c', d_1) = 1$ also $c' \mid c$ so $c = c'$. Thus $d_1 \equiv 1 \pmod{q/c}$. So

$$a' = a_1 a + b_1 c \equiv a_1 a \equiv d_1 a \equiv a \pmod{(c, q/c)}.$$

This proves the set of inequivalent cusps is given by the claimed formula.

To find the width of the cusp $\mathfrak{a} := a/c$ with need to find the generator of $\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} \subset B$.

Let the generator be $\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$. Thus

$$\begin{aligned} \Gamma_{\mathfrak{a}} &= \sigma_{\mathfrak{a}} \Gamma_{\infty} \sigma_{\mathfrak{a}}^{-1} \cap \Gamma_0(q) \\ &= \left\{ \pm \begin{pmatrix} a & * \\ c & * \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \begin{pmatrix} * & * \\ -c & a \end{pmatrix} \in \Gamma_0(q) \right\} \\ &= \left\{ \pm \begin{pmatrix} 1 - mac & ma^2 \\ mc^2 & 1 + mac \end{pmatrix} : q \mid mc^2 \right\}. \end{aligned}$$

So m ranges over all the multiples of $q/(q, c^2)$ and this shows that $m = q/(q, c^2)$. Thus the width of the cusp a/c is $q/(q, c^2)$.

4.9. Exercise. Let \mathfrak{a} and \mathfrak{b} are two cusps for a congruence subgroup Γ with scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$.

(1) Prove the following disjoint decomposition of double cosets:

$$\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} = \delta_{\mathfrak{ab}} B \cup \bigcup_{c > 0} \bigcup_{d \pmod{c}} B \begin{pmatrix} * & * \\ c & d \end{pmatrix} B,$$

where B is the set of upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z})$ and $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ such that it belongs to $\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}$.

(2) Define

$$C(\mathfrak{a}, \mathfrak{b}) := \left\{ c > 0 \mid \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} \right\}.$$

Also define $c(\mathfrak{a}, \mathfrak{b})$ to be the smallest element of $C(\mathfrak{a}, \mathfrak{b})$. Let

$$C := \max c(\mathfrak{a}, \mathfrak{a}), c(\mathfrak{b}), \mathfrak{b}.$$

Prove that for any $X > 0$

$$\sum_{0 < c \leq X} c^{-1} \left| \left\{ d \pmod{c} \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} \right\} \right| \leq C^{-1} X.$$

Hence for any $c \in C(\mathfrak{a}, \mathfrak{b})$ we have

$$\left| \left\{ d \pmod{c} \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} \right\} \right| \leq C^{-1} c^2.$$

(3) Let $z \in \mathbb{H}$ and $Y > 0$. Then prove that

$$|\{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma \mid \Im \sigma_{\mathfrak{a}}^{-1} \gamma z > Y\}|^{-1} \ll \frac{1}{c(\mathfrak{a}, \mathfrak{a}) Y}.$$

4.10. Solution.

(1) First note that for any cusp \mathfrak{a} we have $\sigma_{\mathfrak{a}}^{-1} \mathfrak{a} = \infty$ and $\sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = B$. We examine the set

$$\Omega_{\infty} := \{\omega \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} \mid \omega \infty = \infty\},$$

which consists of the upper-triangular matrices in $\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}$. Suppose that Ω_{∞} is not empty, say $\omega := \sigma_{\mathfrak{a}}^{-1} \gamma \sigma_{\mathfrak{b}} \in \Omega_{\infty}$. Evaluating γ at \mathfrak{b} , we get that

$$\gamma \mathfrak{b} = \sigma_{\mathfrak{a}} \Gamma \sigma_{\mathfrak{b}}^{-1} = \sigma_{\mathfrak{a}} \infty = \mathfrak{a}.$$

Hence \mathfrak{a} and \mathfrak{b} are equivalent; hence they are same cusps and $\gamma \in \Gamma_{\mathfrak{a}}$ and $\omega \in B$. Therefore, $\Omega_{\infty} = B$ if $\mathfrak{a} = \mathfrak{b}$, and empty otherwise.

Let $\omega := \begin{pmatrix} a & * \\ c & d \end{pmatrix}$ be any other element of $\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}$. Then from the relation that

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \begin{pmatrix} a & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} = \begin{pmatrix} a + cm & * \\ c & d + cn \end{pmatrix}$$

we can conclude that the double coset $\Omega := B \begin{pmatrix} * & * \\ c & d \end{pmatrix} B$ determines c uniquely and d modulo a multiple of c . Moreover, given c, d with ω , the double coset Ω does not depend on the upper row of ω . To see that if $\omega := \begin{pmatrix} a' & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}$ then

$$\omega' \omega^{-1} \in \sigma_{\mathfrak{a}}^{-1} \Gamma_{\mathfrak{a}} \sigma_{\mathfrak{a}} = B.$$

Thus $w' \in \omega B$ and so $a' = a + cm$ for some $m \in \mathbb{Z}$. Hence the disjoint decomposition follows.

(2) Let $C = c(\mathfrak{a}) \geq c(\mathfrak{b})$, if not, by symmetry we can interchange the cusps by inversion. If $\omega = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ and $\omega' = \begin{pmatrix} * & * \\ c' & d' \end{pmatrix}$ with $0 < c, c' \leq X$ then $\omega'' := \omega' \omega^{-1} = \begin{pmatrix} * & * \\ c'' & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}}$ with $c'' = c'd - cd'$. If $c'' = 0$ then, as in the previous case, the cusps will be equal. So we assume that $c'' \neq 0$ and thus $|c''| \geq C$ and so

$$\left| \frac{d'}{c'} - \frac{d}{c} \right| \geq \frac{C}{cc'} \geq \frac{C}{cX}.$$

As $(c, d) = 1$ the fraction $\frac{d}{c}$ uniquely determines the pair. Let A be set of $\frac{d}{c}$ in $[0, 1]$ with the prescribed gap. Then

$$\begin{aligned} 1 &\geq \max(A) - \min(A) \\ &\geq \sum_{0 < c \leq x} \sum_{d/c \text{ and } d'/c' \text{ are successive}} \frac{d}{c} - \frac{d'}{c'} \\ &\geq \frac{C}{X} \sum_{0 < c \leq X} c^{-1} \left| \left\{ d \pmod{c} \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_a^{-1} \Gamma \sigma_b \right\} \right|, \end{aligned}$$

which completes the proof. Second part is immediate from the proof.

- (3) Again as in solution 4.6 we may assume that, possibly conjugating, $\mathfrak{a} = \infty$. Let $\gamma \in \Gamma \setminus \Gamma_{\mathfrak{a}}$, so $c > 0$. Also acting by element from Γ_{∞} on the left of γz , which amounts to translation of γz we can also assume that

$$z \in \Gamma_{\infty} \setminus \mathbb{H} \cap D \implies 0 < \Re(z) < 1, |cz + d| > 1.$$

Thus from

$$\Im(\gamma z) = \frac{\Im(z)}{|cz + d|^2} > Y$$

we conclude that $\Im(z) > Y$. Also,

$$c < (\Im(z)Y)^{-1/2}, |c\Re(z) + d| < (\Im(z)/Y)^{1/2}.$$

Thus for $C \leq c < 2C$,

$$|c\Re(z) + d| < \frac{1}{C} (\Im(z)/Y)^{1/2} \implies d/c \in \left[-1 - \frac{1}{C} (\Im(z)/Y)^{1/2}, \frac{1}{C} (\Im(z)/Y)^{1/2}\right].$$

From the spacing property of the possible d/c , as in the previous solution, we conclude that possible number of (c, d) is

$$\ll \frac{C}{c(\infty, \infty)} (\Im(z)/Y)^{1/2}.$$

Now summing over the dyadic intervals with $C = 2^{-n} (\Im(z)Y)^{-1/2}$ we get that number of possible γ which are not in B is

$$\ll \sum_{n=1}^{\infty} 2^{-n} \frac{(\Im(z)Y)^{-1/2}}{c(\infty, \infty)} (\Im(z)/Y)^{1/2} \ll \frac{1}{c(\infty, \infty)Y}.$$

Now adding one more point for $\gamma \in B$ we conclude.