MODULAR FORMS EXERCISES AND SOLUTIONS

1. Due on 26th September

1.1. **Exercise.** Let $\mathcal{P}$ be the set of primes. Prove that $\sum_{p\in\mathcal{P}} \frac{1}{p} = +\infty$.

1.2. **Solution.** Let $s > 1$. Then from the Euler product of the Zeta function,

$$\begin{aligned}
\log\zeta(s) &= \sum_{p\in\mathcal{P}} -\log(1-p^{-s}) = \sum_{p\in\mathcal{P}}\sum_{k=1}^{\infty}\frac{1}{kp^{ks}} \\
&\leq \sum_{p\in\mathcal{P}}\frac{1}{p^s} + \sum_{p\in\mathcal{P}}\sum_{k=2}\frac{1}{p^k} = \sum_{p\in\mathcal{P}}\frac{1}{p^s} + \sum_{p\in\mathcal{P}}\frac{1}{p(p-1)} \\
&= \sum_{p\in\mathcal{P}}\frac{1}{p^s} + O(1)
\end{aligned}$$

As we know that $\lim_{s\to 1+}\zeta(s) = +\infty$, letting $s \to 1+$ in the above inequality we conclude that

$$\lim_{s\to 1+}\sum_{p\in\mathcal{P}}\frac{1}{p^s} = +\infty,$$

hence the result.

1.3. **Summation by Parts.** Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, let $0 < y < x$ and let $f : [y, x] \to \mathbb{C}$ be a function with continuous derivative on $[y, x]$. Then

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where $A(x) = \sum_{n\leq x} a_n$.

1.4. **Exercise.** Prove that for every $\delta > 0$,

$$\pi(x) := |\{p \in \mathcal{P} \mid p \leq x\}|$$

is bigger than $\frac{x}{(\log x)^{1+\delta}}$ for some sufficiently large x.

1.5. **Solution.** Let a_n be the prime indicator function, i.e.

$$a_n := \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not a prime.} \end{cases}.$$

Using summation by parts we note that,

$$\sum_{p \le x} \frac{1}{p} = \sum_{3/2 < n \le x} \frac{a_n}{n} = \frac{\pi(x)}{x} + \int_{3/2}^{x} \frac{\pi(t)}{t^2} dt.$$

If the claim is false i.e. for all sufficiently large x, $\pi(x) \le x/(\log x)^{1+\delta}$ then from the above,

$$\sum_{p \le x} \frac{1}{p} \le \frac{1}{(\log x)^{1+\delta}} + C + \frac{1}{(\log x)^{\delta}},$$

for some constant C. The RHS of the above tends to C as $x \to \infty$ contradicting Exercise 1.1, hence the result.

1.6. **Exercise.** Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\}$ is the fractional part of x. Using this show that $\zeta(s)$ has meromorphic continuation to $\Re(s) > 0$ with a simple pole at $s = 1$.

1.7. **Solution.** Let $\Re(s) > 1$. Then using the summation by parts as following.

$$\sum_{n \le x} \frac{1}{n^s} = \frac{[x]}{x^s} + s \int_1^x \frac{[t]}{t^{s+1}} dt = \frac{1}{x^{s-1}} - \frac{\{x\}}{x^s} + s \int_1^x t^{-s} dt - s \int_1^x \frac{\{t\}}{t^{s+1}} dt$$
$$= \frac{s}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}} dt + O(x^{-\Re(s)} + x^{-\Re(s)+1}).$$

Letting $x \to \infty$, as $\Re(s) > 1$, we conclude that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx.$$

We now note that the integral right hand side is well defined for $\Re(s) > 0$ and is holomorphic in s. As $\frac{s}{s-1}$ is a meromorphic function with simple pole at $s = 1$ and residue 1, we conclude the meromorphic continuation of $\zeta(s)$ to $\Re(s) > 0$.

1.8. **Exercise.** Prove that the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^s \frac{dt}{t}$$

has analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\le 0}$ with simple pole at each non-positive integer. Find the residues of the Gamma function at those poles.
Hint: First prove that $\Gamma(s+1) = s\Gamma(s)$.

1.9. **Solution.** By integration by parts we see that

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1} \frac{dt}{t} = \int_0^\infty e^{-t} s t^s \frac{dt}{t} = s\Gamma(s),$$

for $\Re(s) > 0$. Thus $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ extends definition of $\Gamma(s)$ to $\Re(s) > -1$ meromorphically with pole at $s = 0$ as

$$\lim_{s \to 0+} \int_0^\infty e^{-t} t^s \frac{dt}{t} = +\infty.$$

The pole is simple, as $\lim_{s\to 0} s\Gamma(s) = 1$, and with residue 1. Similarly $\Gamma(s)$ can be extended to all $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple poles at $s = -n$, $n \in \mathbb{N}$ with residue,

$$\lim_{s\to -n} (s+n)\Gamma(s) = \lim_{s \to -n} \frac{\Gamma(s+n+1)}{(s+n-1)\dots s} = \frac{(-1)^n}{n!}.$$

1.10. **Exercise.** Prove the Poisson summation formula: Let $f \in \mathcal{S}(\mathbb{R})$ be in the Schwartz class. Prove that

$$\sum_{n\in\mathbb{Z}} f(n+u) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e(nu).$$

Note: Putting $u = 0$ we get the usual Poisson summation formula.

1.11. **Solution.** Let

$$F(x) : \sum_{n\in\mathbb{Z}} f(n+x)$$

which is a function on $L^1(\mathbb{R}/\mathbb{Z})$ so has a Fourier expansion of the form

$$F(x) = \sum_{n\in\mathbb{Z}} e(nx)\hat{F}(n).$$

Here

$$\begin{aligned}\hat{F}(n) &= \int_0^1 F(x) e(-nx) dx = \sum_{n\in\mathbb{Z}} \int_0^1 \sum_{m\in\mathbb{Z}} f(m+x) e(-nx) dx \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(x) e(-nx) = \int_{-\infty}^\infty f(x) e(-nx) dx = \hat{f}(n),\end{aligned}$$

this provides the result.

1.12. **Exercise.** Recall that,

$$G(1, N) := \sum_{n \mod N} e(n^2/N).$$

Prove that

(1) For any odd positive integer N, $G(1, N^2) = N$ and $G(1, N^3) = NG(1, N)$.
(2) For every positive integer N, $G(1, N) = \frac{1+i^{-N}}{1-i}\sqrt{N}$.

1.13. **Solution.** (1) is elementary. We can parametrize the residue class of N^k by

$$\{a_1N^{k-1} + a_2N^{k-2} + \cdots + a_k \mid 0 \le a_i \le N-1\}.$$

Using this we have,

$$\begin{aligned} G(1, N^2) &= \sum_{a=0}^{N-1}\sum_{b=0}^{N-1} e\left(\frac{(aN+b)^2}{N^2}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \sum_{a=0}^{N-1} e\left(\frac{2ab}{N}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2)\delta_{b=0}N = N. \end{aligned}$$

Similarly,

$$\begin{aligned} G(1, N^3) &= \sum_{a=0}^{N-1}\sum_{b=0}^{N-1}\sum_{c=0}^{N-1} e\left(\frac{(aN^2+bN+c)^2}{N^3}\right) \\ &= \sum_{c=0}^{N-1}\sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) \sum_{a=0}^{N-1} e(2ac/N) \\ &= \sum_{c=0}^{N-1}\sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) N\delta_{c=0} = NG(1, N). \end{aligned}$$

For the second part we use the Poisson summation formula. First we note the function

$$f(x) := 1_{[0,N]}e(x^2/N)$$

is a function which is continuous on $(0, N)$ and has continuity only from one side at $x = 0, N$. From the Fourier theory we know that the Fourier series of f at $x = 0$ would converge to $\frac{f(0+)+f(0-)}{2} = f(0+)/2$. and similarly, at $x = N$ to $f(N-)/2$ Thus using the (modified) Poisson summation formula and using that $f(0+) = f(N-)$ we get that,

$$\begin{aligned} \sum_{n=0}^{N} e(N^2/N) &= \frac{f(0+)}{2} + \sum_{n=1}^{N-1} f(n) + \frac{f(N-)}{2} \\ &= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e(nx)dx = \sum_{n\in\mathbb{Z}} \int_0^N e(x^2/N + nx)dx. \end{aligned}$$

Thus,

$$G(1, N) = N\sum_{n\in\mathbb{Z}} \int_0^1 e(Nx^2 + nNx)dx = N\sum_{n\in\mathbb{Z}} e(-Nn^2/4)\int_0^1 e(N(x+n/2)^2)$$

Noting that

$$e(-Nn^2/4) = \begin{cases} 1, \text{ if } n \text{ is even,} \\ i^{-N}, \text{ if } n \text{ is odd.} \end{cases},$$

and dividing the above sum into odd and even parts we get that,

$$\begin{aligned} G(1,N) &= N \sum_{n\in\mathbb{Z}} \int_{n}^{1+n} e(Nx^2)dx + Ni^{-N} \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} e(nx^2)dx \\ &= \sqrt{N}(1+i^{-N}) \int_{-\infty}^{\infty} e(y^2)dy. \end{aligned}$$

The last integral can be checked convergent and we call it C. Thus,

$$G(1,N) = \sqrt{N}C(1+i^{-N}).$$

Checking that, $G(1,1) = 1$, we conclude the result.

1.14. **Dirichlet Character.** A *Dirichlet character with modulus* q is a character

$$\chi : \mathbb{Z}/q\mathbb{Z}^\times \to \mathbb{C}^\times$$

extended to $\mathbb{Z}$ by making it q-periodic and defining $\chi(a) = 0$ for $(a,q) > 1$. Associated to each character χ, in addition to its modulus q, is a natural number q', its conductor. The *conductor* q' is the smallest divisor of q such that χ can be written as $\chi = \chi'\chi_0$, where χ_0 is the trivial Dirichlet character mod q and χ' is a character of modulus q'. If a character has conductor equal to to its modulus then it is called a *primitive Dirichlet character.* Check that, for a primitive Dirichlet character χ mod q one has

$$\frac{1}{q} \sum_{a \mod q} \chi(ma+b) = \begin{cases} \chi(b), \text{ if } q \mid m \\ 0, \text{ if } q \nmid m. \end{cases}$$

The above is not true for a non-primitive character.

1.15. **Exercise.** Let χ be a primitive Dirichlet character mod q and $f \in L^1(\mathbb{R})$. Prove that

$$\sum_{m\in\mathbb{Z}} f(m)\chi(m) = \frac{G(\chi)}{q} \sum_{n\in\mathbb{Z}} \hat{f}(n/q)\bar{\chi}(n),$$

where $G(\chi)$ is the Gauss sum attached to χ defined by

$$G(\chi) := \sum_{a \mod q} \chi(a)e(a/q).$$

Hint: Use the Poisson summation formula.

1.16. **Solution.** First we prove the following. Let $v \in \mathbb{R}$ and $u \in \mathbb{R}^+$. Then using the Poisson summation formula,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} f(um+v) &= \sum_{m\in\mathbb{Z}} \int_{\infty}^{\infty} f(ux+v)e(-mx)dx \\ &= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e(-m(x-v)/u)\frac{dx}{u} \\ &= \frac{1}{u}\sum_{m\in\mathbb{Z}} \hat{f}(m)e(mv/u).\end{aligned}$$

Using the above we get that,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} f(m)\chi(m) &= \sum_{m\in\mathbb{Z}} \sum_{a \mod q} \chi(a)f(mq+a) \\ &= \sum_{a \mod q} \chi(a)\frac{1}{q}\sum_{m\in\mathbb{Z}} \hat{f}(m)e(ma/q) \\ &= \frac{G(\chi)}{q}\sum_{m\in\mathbb{Z}} \hat{f}(m)\bar{\chi}(m).\end{aligned}$$

Here in the last line we have used that for a primitive Dirichlet character χ,

$$\sum_{a \mod q} \chi(a)e(am/q) = \bar{\chi}(m)G(\chi).$$

This can be seen as follows. Let $(m,q)=1$. Then,

$$\bar{\chi}(m)G(\chi) = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(am^{-1})e(a/q) = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(a)e(am/q).$$

If $(m,q)>1$ then it follows from the fact that $\chi(m)=0$ and

$$\sum_{a \mod q} \chi(a)e(am/q) = \sum_{y \mod q/(q,m)} e(ym/q) \sum_{x \mod q} \chi(xq/d+y) = 0.$$

2. Due on 10th October

2.1. **Exercise.** Prove that $\Gamma(q)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and has index in it $q^3 \prod_{p|q}(1-\frac{1}{p^2})$.

2.2. **Solution.** We consider the $\mod q$ reduction map

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}),$$

whose kernel is by definition $\Gamma(q)$. Thus $\Gamma(q)$ is normal. Hence, as the above map is surjective, by the first isomorphism theorem

$$\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(q),$$

and so,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)] = |\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})|.$$

To compute the cardinality we first note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ then $(c, d, q) = 1$. For each such lower row (c, d) we have exactly q solutions for the congruence $ad - bc \equiv 1 \mod q$. Thus the cardinality is,

$$q|\{(c, d) \mod q \mid (c, d, q) = 1\}| = q \sum_{r|q} \mu(r)(q/r)^2 = q^3 \prod_{p|q}(1 - p^{-2}).$$

2.3. **Exercise.** Recall the subgroups $\Gamma_0(q)$, $\Gamma_1(q)$ and $\Gamma_d(q)$ of $\mathrm{SL}_2(\mathbb{Z})$ from the lectures. Compute indices of the subgroups in $\mathrm{SL}_2(\mathbb{Z})$.

2.4. **Solution.** Consider the surjective map

$$\Gamma_1(q) \to \mathbb{Z}/q\mathbb{Z},$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \mod q.$$

The kernel of this map is by definition $\Gamma(q)$. Thus by the first isomorphism theorem,

$$\Gamma_1(q)/\Gamma(q) \cong \mathbb{Z}/q\mathbb{Z}.$$

Hence,

$$[\mathrm{SL}_2(Z) : \Gamma_1(q)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q][\Gamma_1(q) : \Gamma(q)]^{-1} = q^2 \prod_{p|q}(1 - p^{-2}).$$

Similarly, considering the map

$$\Gamma_0(q) \to (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod q,$$

we conclude that

$$\Gamma_0(q)/\Gamma_1(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(Z):\Gamma_1(q)] = \frac{1}{\phi(q)} q^2 \prod_{p|q}(1-p^{-2}) = q\prod_{p|q}(1+p^{-1}).$$

Again similarly, considering the map

$$\Gamma_d(q) \to (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod q,$$

we conclude that

$$\Gamma_d(q)/\Gamma(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(Z):\Gamma_d(q)]\frac{1}{\phi(q)} q^3 \prod_{p|q}(1-p^{-2}) = q^2\prod_{p|q}(1+p^{-1}).$$

2.5. **Exercise.** Prove that for any finite abelian group G one has $G \cong \hat{G}$.
Hint: First try to show for finite abelian groups G_1 and G_2 that $\hat{G}_1 \times \hat{G}_2 \cong \widehat{G_1 \times G_2}$. Then use the structure theory of the finite abelian groups.

2.6. **Solution.** We define a map

$$\hat{G}_1 \times \hat{G}_2 \to \widehat{G_1 \times G_2} \text{ by } (\chi_1, \chi_2) \mapsto \{\chi : (g_1, g_2) \mapsto \chi_1(g_1)\chi_2(g_2)\}.$$

This map is clearly well-defined homomorphism. To see injectivity if χ is the trivial character then

$$\chi_1(g_1) = \chi_2^{-1}(g_2) \forall (g_1, g_2) \in G_1 \times G_2,$$

which implies that χ_i are the trivial character. From the lecture we recall that $|G|= |\hat{G}|$, which proves the isomorphism. Now from the structure theory of the finite abelian groups we know that every finite abelian group is isomorphic to direct product of $\mathbb{Z}/_n\mathbb{Z}$. hence it is enough to show that

$$\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \cong \mu_n \cong \mathbb{Z}/n\mathbb{Z},$$

where μ_n is the group of n'th roots of unity. To See this isomorphism we consider that map

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \to \mu_n \text{ by } \chi \mapsto \chi(1).$$

This map is clearly a well-defined homomorphism, as $\chi(1)^n = \chi(n) = \chi(0) = 1$, i.e. $\chi(1) \in \mu_n$. If $\chi(1) = 1$ then $\chi(m) = \chi^m(1) = 1$, which proves the injectivity. Equality of the cardinalities concludes the proof.

2.7. **Exercise.** Recall the the product expansion

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(1) Use the above formula to prove that,

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left[\frac{1}{z-d} + \frac{1}{z+d}\right] = \pi \cot(\pi z) = \pi i - 2\pi i \sum_{d=0}^{\infty} e(dz).$$

(2) Prove that for even natural number k

$$\zeta(k) = -\frac{(2\pi i)^k}{2k!} B_k,$$

where B_k are the Bernoulli numbers.

(3) Prove that $\zeta(s)$ has zeros at negative even integers.
Hint: Use the functional equation of $\zeta(s)$.

2.8. **Solution.**

(1) We do a logarithmic differentiation of the given expression.

$$\begin{aligned}\pi \cot(\pi z) &= \frac{d}{dz} \log \sin(\pi z) \\ &= \frac{d}{dz} \log(\pi z) + \frac{d}{dz} \sum_{n=1}^{\infty} \log(1 - z^2/n^2) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2},\end{aligned}$$

hence the first equality. For the second equality we see that,

$$\pi \cot(\pi z) = \pi i \frac{e(z)+1}{e(z)-1} = \pi i - 2\pi i \frac{1}{1-e(z)} = \pi i - 2\pi i \sum_{n=0}^{\infty} e(nz),$$

completing the proof.

(2) Recall that the Bernoulli numbers are defined by the coefficient of the series expansion of $\frac{x}{e^x - 1}$, i.e.

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.$$

Consider the generating series of $\zeta(2k)$

$$1 + 2\sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

For $|z|< 1$ the above sum is absolutely convergent, so plugging in the definition of $\zeta(s)$ for $s > 1$ and changing the order of the summation we get that above sum is

$$1+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(z/n)^{2k} = 1+\sum_{n=1}^{\infty}\frac{2z^2}{n^2-z^2} = \pi z\cot(\pi z),$$

where the last equality is from (1). But from (2)

$$\pi z\cot(\pi z) = \pi i z - \frac{2\pi i z}{1-e^{2\pi i z}} = \pi i z - \sum_{k=0}^{\infty} B_k\frac{(2\pi i z)^k}{k!}.$$

Equating two power series we conclude that

$$2\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!},$$

concluding the result.

(3) We recall the functional equation of $\zeta(s)$

$$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \zeta(1-s)\pi^{(1-s)/2}\Gamma((1-s)/2).$$

We also recall the duplication formula,

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma(s/2)\Gamma((1+s)/2),$$

and

$$\Gamma(1/2-s/2)\Gamma(1/2+s/2) = \frac{\pi}{\cos(\pi s/2)}.$$

Combining all of them we get that,

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s).$$

Plugging in $s = 2n+1$ for $n \geq 1$ and checking that $\cos(n\pi+\pi/2) = 0$ we conclude that

$$\zeta(-2n) = 0.$$

2.9. **Eisenstein Series of weight 2.** In the lecture we have defined Eisenstein series E_k of weight k for $k > 2$. In this exercise we will define *Eisenstein series* E_2 *of weight* 2 and will show that it satisfies an "almost modularity" relation.

2.10. **Exercise.** Define the following functions for $z \in \mathbb{H}$:

$$G_2(z) := \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2},$$

$$G_2^*(z) := G_2(z) - \frac{\pi}{2\Im(z)},$$

$$G_{2,\epsilon} := \frac{1}{2}\sum_{(m,n)\neq(0,0)}\frac{1}{(mz+n)^2}\frac{1}{|mz+n|^{2\epsilon}},\ \text{for } \epsilon > 0.$$

(1) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove that $G_{2,\epsilon}$ converges absolutely and locally uniformly. Also show that,

$$G_{2,\epsilon}(\gamma z) = (cz+d)^2 |cz+d|^{2\epsilon} G_{2,\epsilon}(z).$$

(2) For $\epsilon > -1/2$ define:

$$I_\epsilon(z) := \int_{\mathbb{R}} \frac{dt}{(z+t)^2 |z+t|^{2\epsilon}} \text{ and } I(\epsilon) := \int_{\mathbb{R}} \frac{dt}{(i+t)^2 (1+t^2)^\epsilon}.$$

Consider

$$G_{2,\epsilon}(z) - \sum_{m=1}^{\infty} I_\epsilon(mz).$$

Use the mean value theorem to prove that it converges absolutely and locally uniformly for $\epsilon > -1/2$ and the limit as $\epsilon \to 0$ is $G_2(z)$.

(3) Show that

$$I_\epsilon(z) = \frac{I(\epsilon)}{\Im(z)^{1+2\epsilon}} \text{ and } I'(0) = -\pi.$$

Use this to show that the limit of $G_{2,\epsilon}(z)$ as $\epsilon \to 0$ is $G_2^*(z)$. Hence G_2^* transforms like a modular form of weight 2.

(4) Conclude that

$$G_2(\gamma z) = (cz+d)^2 G_2(z) - \pi i c(cz+d).$$

E_2 is defined to be, as usual, $\frac{G_2}{\zeta(2)}$.

2.11. **Solution.**

(1) Note that, for $k > 2$ and $z \in \mathbb{H}$

$$\sum_{N=1}^{\infty} \sum_{N < |mz+n| \leq N+1} \frac{1}{|mz+n|^k} \leq \sum_{N=1}^{\infty} \frac{\#\{(m,n) \in \mathbb{Z}^2 \mid N \leq |mz+n| \leq N+1\}}{N^k}.$$

It is easy to check that

$$\#\{(m,n) \mid N \leq |mz+n| \leq N+1\} \ll \pi(N+1)^2 - \pi N^2 \ll N.$$

Thus the above sum is, as $k > 2$

$$\ll \sum_{N=1}^{\infty} N^{1-k} < \infty.$$

Now we see that,

$$G_{2,\epsilon} \leq \sum_{0 \leq |mz+n| \leq 1} |mz+n|^{-2-2\epsilon} + \sum_{1 \leq |mz+n|} |mz+n|^{-2-2\epsilon}.$$

The first sum has finite number of summands and second sum is absolutely and locally uniformly convergent by the previous argument. Thus the sum of $G_{2,\epsilon}$ are convergent abolustely and locally uniformly, thus defines a holomorphic function on $\mathbb{H}$. To see the transformation law we first note that every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ induces a bijection from $\mathbb{Z}^2 \setminus \{(0,0)\}$ to itself by right multiplication. Also one checks that,

$$m\gamma z + n = \frac{(ma+nc)z + (mb+nd)}{cz+d} = \frac{m'z+n'}{cz+d}.$$

Combining these two facts, we conclude that

$$G_2, \epsilon(\gamma z) = \sum_{(m',n')\neq(0,0)} \frac{(cz+d)^2|cz+d|^{2\epsilon}}{(m'z+n')|m'z+n'|^{2\epsilon}} = (cz+d)^2|cz+d|^{2\epsilon}G_{2,\epsilon}(z).$$

(2) Let

$$f(t) := (mz+t)^2|mz+t|^{-2\epsilon},$$

with implicit dependence on mz. Now as we have proved the absolute convergence of the $\sum f(n)$ we will freely change the order of summation and order of integration and summation, as follows.

$$\begin{aligned}
\tilde{G}_{2,\epsilon}(z) &= G_{2,\epsilon}(z) - \sum_{m=0}^{\infty} I_\epsilon(mz) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} (f(n) - \int_n^{n+1} f(t)dt) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \int_n^{n+1} (f(n) - f(t))dt.
\end{aligned}$$

By the mean value theorem on $n \le t \le n+1$ we get that

$$|f(n) - f(t)| \le \sup_{n\le u\le n+1} |f'(u)| \ll |mz+n|^{-3-2\epsilon}.$$

Hence, the sum is absolutely convergent for $\epsilon > -1/2$ and thus $\lim_{\epsilon\to 0}\tilde{G}_{2,\epsilon}$ exists and defines a holomorphic function. We calculate,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\tilde{G}_{2,\epsilon}(z) \\
&= \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \sum_{m=1}^{\infty}\left[\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2} + \sum_{n\in\mathbb{Z}}\left(\frac{1}{mz+n+1} - \frac{1}{mz+n}\right)\right] \\
&= \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2} \\
&= G_2(z)
\end{aligned}$$

(3) Let $z = x + iy$. Then changing variable $t \mapsto yt - x$ we get that,

$$I_\epsilon(x+iy) = \int_{\mathbb{R}} \frac{dt}{(x+t+iy)^2|x+t+iy|^{2\epsilon}}$$
$$= \frac{1}{y^{1+2\epsilon}} \int_{\mathbb{R}} \frac{dt}{(t+i)^2|t+i|^{2\epsilon}} = \frac{I(\epsilon)}{y^{1+2\epsilon}}.$$

Differentiating under the integration sign and then integrating by parts we get that,

$$I'(0) = -\int_{\mathbb{R}} \frac{\log(1+t^2)}{(t+i)^2} dt = \frac{\log(1+t^2)}{t+i}\bigg|_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{2tdt}{(t+i)(1+t^2)}$$
$$= -\int_{\mathbb{R}} \frac{1}{(t+i)^2} + \frac{1}{1+t^2} = -\int_{\mathbb{R}} \frac{dt}{t^2+1} = -\pi.$$

Using the above two results we compute that,

$$\lim_{\epsilon\to 0} \sum_{m=1}^{\infty} I_\epsilon(mz) = \lim_{\epsilon\to 0} \sum_{m=1}^{\infty} \frac{I(\epsilon)}{(my)^{1+2\epsilon}} = \lim_{\epsilon\to 0} \frac{I(\epsilon)\zeta(1+2\epsilon)}{\Im(z)^{1+2\epsilon}}.$$

From the exercise 1.6 we know that

$$\zeta(1+2\epsilon) = \frac{1}{2\epsilon} + O(1).$$

Using that $I(0) = 0$ we have that above limit equals to

$$\lim_{\epsilon\to 0} \frac{I(\epsilon)}{2\epsilon\Im(z)^{1+2\epsilon}} = \frac{I'(0)}{2\Im(z)}.$$

Thus,

$$\lim_{\epsilon\to 0} G_{2,\epsilon}(Z) = \lim_{\epsilon\to 0}\left(\tilde{G}_{2,\epsilon}(z) + \sum_{m=1}^{\infty} I_\epsilon(mz)\right) = G_2(z) - \frac{\pi}{2\Im(z)} = G_2^*(z).$$

(4) From part (1) and (3) letting $\epsilon \to 0$ we see that $G_2^*(z)$ transforms as a modular form of weight 2. So,

$$G_2(\gamma z) - (cz+d)^2 G_2(z) = \frac{\pi}{2\Im(\gamma z)} - (cz+d)^2 \frac{\pi}{2\Im(z)}$$
$$= \frac{\pi}{2\Im(z)}(|cz+d|^2 - (cz+d)^2)$$
$$= \pi i c(cz+d),$$

concluding the result.