MODULAR FORMS EXERCISES AND SOLUTIONS

1. Due on 26th September

1.1. Exercise. Let $\mathcal{P}$ be the set of primes. Prove that $\sum_{p\in\mathcal{P}} \frac{1}{p} = +\infty$.

1.2. Solution. Let $s > 1$. Then from the Euler product of the Zeta function,

$$\begin{aligned}\log\zeta(s) &= \sum_{p\in\mathcal{P}} -\log(1-p^{-s}) = \sum_{p\in\mathcal{P}}\sum_{k=1}^{\infty}\frac{1}{kp^{ks}} \\ &\leq \sum_{p\in\mathcal{P}}\frac{1}{p^s} + \sum_{p\in\mathcal{P}}\sum_{k=2}\frac{1}{p^k} = \sum_{p\in\mathcal{P}}\frac{1}{p^s} + \sum_{p\in\mathcal{P}}\frac{1}{p(p-1)} \\ &= \sum_{p\in\mathcal{P}}\frac{1}{p^s} + O(1)\end{aligned}$$

As we know that $\lim_{s\to 1+}\zeta(s) = +\infty$, letting $s \to 1+$ in the above inequality we conclude that

$$\lim_{s\to 1+}\sum_{p\in\mathcal{P}}\frac{1}{p^s} = +\infty,$$

hence the result.

1.3. Summation by Parts. Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, let $0 < y < x$ and let $f : [y, x] \to \mathbb{C}$ be a function with continuous derivative on $[y, x]$. Then

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where $A(x) = \sum_{n\leq x} a_n$.

1.4. Exercise. Prove that for every $\delta > 0$,

$$\pi(x) := |\{p \in \mathcal{P} \mid p \leq x\}|$$

is bigger than $\frac{x}{(\log x)^{1+\delta}}$ for some sufficiently large x.

1.5. **Solution.** Let a_n be the prime indicator function, i.e.

$$a_n := \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not a prime.} \end{cases}.$$

Using summation by parts we note that,

$$\sum_{p \le x} \frac{1}{p} = \sum_{3/2 < n \le x} \frac{a_n}{n} = \frac{\pi(x)}{x} + \int_{3/2}^{x} \frac{\pi(t)}{t^2} dt.$$

If the claim is false i.e. for all sufficiently large x, $\pi(x) \le x/(\log x)^{1+\delta}$ then from the above,

$$\sum_{p \le x} \frac{1}{p} \le \frac{1}{(\log x)^{1+\delta}} + C + \frac{1}{(\log x)^{\delta}},$$

for some constant C. The RHS of the above tends to C as $x \to \infty$ contradicting Exercise 1.1, hence the result.

1.6. **Exercise.** Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\}$ is the fractional part of x. Using this show that $\zeta(s)$ has meromorphic continuation to $\Re(s) > 0$ with a simple pole at $s = 1$.

1.7. **Solution.** Let $\Re(s) > 1$. Then using the summation by parts as following.

$$\sum_{n \le x} \frac{1}{n^s} = \frac{[x]}{x^s} + s \int_1^x \frac{[t]}{t^{s+1}} dt = \frac{1}{x^{s-1}} - \frac{\{x\}}{x^s} + s \int_1^x t^{-s} dt - s \int_1^x \frac{\{t\}}{t^{s+1}} dt$$

$$= \frac{s}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}} dt + O(x^{-\Re(s)} + x^{-\Re(s)+1}).$$

Letting $x \to \infty$, as $\Re(s) > 1$, we conclude that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$

We now note that the integral right hand side is well defined for $\Re(s) > 0$ and is holomorphic in s. As $\frac{s}{s-1}$ is a meromorphic function with simple pole at $s = 1$ and residue 1, we conclude the meromorphic continuation of $\zeta(s)$ to $\Re(s) > 0$.

1.8. **Exercise.** Prove that the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) := \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

has analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\le 0}$ with simple pole at each non-positive integer. Find the residues of the Gamma function at those poles.
Hint: First prove that $\Gamma(s+1) = s\Gamma(s)$.

1.9. **Solution.** By integration by parts we see that

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1} \frac{dt}{t} = \int_0^\infty e^{-t} s t^s \frac{dt}{t} = s\Gamma(s),$$

for $\Re(s) > 0$. Thus $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ extends definition of $\Gamma(s)$ to $\Re(s) > -1$ meromorphically with pole at $s = 0$ as

$$\lim_{s \to 0+} \int_0^\infty e^{-t} t^s \frac{dt}{t} = +\infty.$$

The pole is simple, as $\lim_{s\to 0} s\Gamma(s) = 1$, and with residue 1. Similarly $\Gamma(s)$ can be extended to all $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple poles at $s = -n$, $n \in \mathbb{N}$ with residue,

$$\lim_{s \to -n} (s+n)\Gamma(s) = \lim_{s \to -n} \frac{\Gamma(s+n+1)}{(s+n-1)\dots s} = \frac{(-1)^n}{n!}.$$

1.10. **Exercise.** Prove the Poisson summation formula: Let $f \in \mathcal{S}(\mathbb{R})$ be in the Schwartz class. Prove that

$$\sum_{n\in\mathbb{Z}} f(n+u) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e(nu).$$

Note: Putting $u = 0$ we get the usual Poisson summation formula.

1.11. **Solution.** Let

$$F(x) : \sum_{n\in\mathbb{Z}} f(n+x)$$

which is a function on $L^1(\mathbb{R}/\mathbb{Z})$ so has a Fourier expansion of the form

$$F(x) = \sum_{n\in\mathbb{Z}} e(nx)\hat{F}(n).$$

Here

$$\begin{aligned}\hat{F}(n) &= \int_0^1 F(x)e(-nx)dx = \sum_{n\in\mathbb{Z}} \int_0^1 \sum_{m\in\mathbb{Z}} f(m+x)e(-nx)dx \\ &= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(x)e(-nx) = \int_{-\infty}^{\infty} f(x)e(-nx)dx = \hat{f}(n),\end{aligned}$$

this provides the result.

1.12. **Exercise.** Recall that,

$$G(1,N) := \sum_{n \mod N} e(n^2/N).$$

Prove that

(1) For any odd positive integer N, $G(1,N^2) = N$ and $G(1,N^3) = NG(1,N)$.
(2) For every positive integer N, $G(1,N) = \frac{1+i^{-N}}{1-i}\sqrt{N}$.

1.13. **Solution.** (1) is elementary. We can parametrize the residue class of N^k by

$$\{a_1N^{k-1} + a_2N^{k-2} + \cdots + a_k \mid 0 \le a_i \le N-1\}.$$

Using this we have,

$$\begin{aligned} G(1, N^2) &= \sum_{a=0}^{N-1}\sum_{b=0}^{N-1} e\left(\frac{(aN+b)^2}{N^2}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \sum_{a=0}^{N-1} e\left(\frac{2ab}{N}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2)\delta_{b=0}N = N. \end{aligned}$$

Similarly,

$$\begin{aligned} G(1, N^3) &= \sum_{a=0}^{N-1}\sum_{b=0}^{N-1}\sum_{c=0}^{N-1} e\left(\frac{(aN^2+bN+c)^2}{N^3}\right) \\ &= \sum_{c=0}^{N-1}\sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) \sum_{a=0}^{N-1} e(2ac/N) \\ &= \sum_{c=0}^{N-1}\sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) N\delta_{c=0} = NG(1, N). \end{aligned}$$

For the second part we use the Poisson summation formula. First we note the function

$$f(x) := 1_{[0,N]}e(x^2/N)$$

is a function which is continuous on $(0, N)$ and has continuity only from one side at $x = 0, N$. From the Fourier theory we know that the Fourier series of f at $x = 0$ would converge to $\frac{f(0+)+f(0-)}{2} = f(0+)/2$. and similarly, at $x = N$ to $f(N-)/2$ Thus using the (modified) Poisson summation formula and using that $f(0+) = f(N-)$ we get that,

$$\begin{aligned} \sum_{n=0}^{N} e(N^2/N) &= \frac{f(0+)}{2} + \sum_{n=1}^{N-1} f(n) + \frac{f(N-)}{2} \\ &= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e(nx)dx = \sum_{n\in\mathbb{Z}} \int_0^N e(x^2/N + nx)dx. \end{aligned}$$

Thus,

$$G(1, N) = N\sum_{n\in\mathbb{Z}} \int_0^1 e(Nx^2 + nNx)dx = N\sum_{n\in\mathbb{Z}} e(-Nn^2/4)\int_0^1 e(N(x+n/2)^2)$$

Noting that

$$e(-Nn^2/4) = \begin{cases} 1, \text{ if } n \text{ is even}, \\ i^{-N}, \text{ if } n \text{ is odd.} \end{cases},$$

and dividing the above sum into odd and even parts we get that,

$$\begin{aligned} G(1,N) &= N \sum_{n \in \mathbb{Z}} \int_{n}^{1+n} e(Nx^2)dx + Ni^{-N} \sum_{n \in \mathbb{Z}} \int_{n-1/2}^{n+1/2} e(nx^2)dx \\ &= \sqrt{N}(1+i^{-N}) \int_{-\infty}^{\infty} e(y^2)dy. \end{aligned}$$

The last integral can be checked convergent and we call it C. Thus,

$$G(1,N) = \sqrt{N}C(1+i^{-N}).$$

Checking that, $G(1,1)=1$, we conclude the result.

1.14. **Dirichlet Character.** A *Dirichlet character with modulus* q is a character

$$\chi : \mathbb{Z}/q\mathbb{Z}^{\times} \to \mathbb{C}^{\times}$$

extended to $\mathbb{Z}$ by making it q-periodic and defining $\chi(a) = 0$ for $(a,q) > 1$. Associated to each character χ, in addition to its modulus q, is a natural number q', its conductor. The *conductor* q' is the smallest divisor of q such that χ can be written as $\chi = \chi'\chi_0$, where χ_0 is the trivial Dirichlet character mod q and χ' is a character of modulus q'. If a character has conductor equal to to its modulus then it is called a *primitive Dirichlet character.* Check that, for a primitive Dirichlet character χ mod q one has

$$\frac{1}{q} \sum_{a \mod q} \chi(ma+b) = \begin{cases} \chi(b), \text{ if } q \mid m \\ 0, \text{ if } q \nmid m. \end{cases}$$

The above is not true for a non-primitive character.

1.15. **Exercise.** Let χ be a primitive Dirichlet character mod q and $f \in L^1(\mathbb{R})$. Prove that

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \frac{G(\chi)}{q} \sum_{n \in \mathbb{Z}} \hat{f}(n/q)\bar{\chi}(n),$$

where $G(\chi)$ is the Gauss sum attached to χ defined by

$$G(\chi) := \sum_{a \mod q} \chi(a)e(a/q).$$

Hint: Use the Poisson summation formula.

1.16. **Solution.** First we prove the following. Let $v \in \mathbb{R}$ and $u \in \mathbb{R}^+$. Then using the Poisson summation formula,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} f(um+v) &= \sum_{m\in\mathbb{Z}} \int_{\infty}^{\infty} f(ux+v)e(-mx)dx \\ &= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e(-m(x-v)/u)\frac{dx}{u} \\ &= \frac{1}{u}\sum_{m\in\mathbb{Z}} \hat{f}(m)e(mv/u).\end{aligned}$$

Using the above we get that,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} f(m)\chi(m) &= \sum_{m\in\mathbb{Z}} \sum_{a \mod q} \chi(a)f(mq+a) \\ &= \sum_{a \mod q} \chi(a)\frac{1}{q}\sum_{m\in\mathbb{Z}} \hat{f}(m)e(ma/q) \\ &= \frac{G(\chi)}{q}\sum_{m\in\mathbb{Z}} \hat{f}(m)\bar{\chi}(m).\end{aligned}$$

Here in the last line we have used that for a primitive Dirichlet character χ,

$$\sum_{a \mod q} \chi(a)e(am/q) = \bar{\chi}(m)G(\chi).$$

This can be seen as follows. Let $(m,q)=1$. Then,

$$\bar{\chi}(m)G(\chi) = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(am^{-1})e(a/q) = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(a)e(am/q).$$

If $(m,q) > 1$ then it follows from the fact that $\chi(m) = 0$ and

$$\sum_{a \mod q} \chi(a)e(am/q) = \sum_{y \mod q/(q,m)} e(ym/q) \sum_{x \mod q} \chi(xq/d+y) = 0.$$

2. Due on 10th October

2.1. **Exercise.** Prove that $\Gamma(q)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and has index in it $q^3\prod_{p|q}(1-\frac{1}{p^2})$.

2.2. **Solution.** We consider the $\mod q$ reduction map

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}),$$

whose kernel is by definition $\Gamma(q)$. Thus $\Gamma(q)$ is normal. Hence, as the above map is surjective, by the first isomorphism theorem

$$\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(q),$$

and so,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)] = |\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})|.$$

To compute the cardinality we first note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ then $(c, d, q) = 1$. For each such lower row (c, d) we have exactly q solutions for the congruence $ad - bc \equiv 1 \mod q$. Thus the cardinality is,

$$q|\{(c, d) \mod q \mid (c, d, q) = 1\}| = q\sum_{r|q}\mu(r)(q/r)^2 = q^3\prod_{p|q}(1 - p^{-2}).$$

2.3. **Exercise.** Recall the subgroups $\Gamma_0(q)$, $\Gamma_1(q)$ and $\Gamma_d(q)$ of $\mathrm{SL}_2(\mathbb{Z})$ from the lectures. Compute indices of the subgroups in $\mathrm{SL}_2(\mathbb{Z})$.

2.4. **Solution.** Consider the surjective map

$$\Gamma_1(q) \to \mathbb{Z}/q\mathbb{Z},$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \mod q.$$

The kernel of this map is by definition $\Gamma(q)$. Thus by the first isomorphism theorem,

$$\Gamma_1(q)/\Gamma(q) \cong \mathbb{Z}/q\mathbb{Z}.$$

Hence,

$$[\mathrm{SL}_2(Z) : \Gamma_1(q)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q][\Gamma_1(q) : \Gamma(q)]^{-1} = q^2\prod_{p|q}(1 - p^{-2}).$$

Similarly, considering the map

$$\Gamma_0(q) \to (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod q,$$

we conclude that

$$\Gamma_0(q)/\Gamma_1(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(Z):\Gamma_1(q)]=\frac{1}{\phi(q)}q^2\prod_{p|q}(1-p^{-2})=q\prod_{p|q}(1+p^{-1}).$$

Again similarly, considering the map

$$\Gamma_d(q)\to(\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mapsto d \mod q,$$

we conclude that

$$\Gamma_d(q)/\Gamma(q)\cong(\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(Z):\Gamma_d(q)]\frac{1}{\phi(q)}q^3\prod_{p|q}(1-p^{-2})=q^2\prod_{p|q}(1+p^{-1}).$$

2.5. **Exercise.** Prove that for any finite abelian group G one has $G\cong\hat{G}$.
Hint: First try to show for finite abelian groups G_1 and G_2 that $\hat{G}_1\times\hat{G}_2\cong\widehat{G_1\times G_2}$. Then use the structure theory of the finite abelian groups.

2.6. **Solution.** We define a map

$$\hat{G}_1\times\hat{G}_2\to\widehat{G_1\times G_2} \text{ by } (\chi_1,\chi_2)\mapsto\{\chi:(g_1,g_2)\mapsto\chi_1(g_1)\chi_2(g_2)\}.$$

This map is clearly well-defined homomorphism. To see injectivity if χ is the trivial character then

$$\chi_1(g_1)=\chi_2^{-1}(g_2)\forall(g_1,g_2)\in G_1\times G_2,$$

which implies that χ_i are the trivial character. From the lecture we recall that $|G|=|\hat{G}|$, which proves the isomorphism. Now from the structure theory of the finite abelian groups we know that every finite abelian group is isomorphic to direct product of $\mathbb{Z}/_n\mathbb{Z}$. hence it is enough to show that

$$\widehat{\mathbb{Z}/n\mathbb{Z}}=\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},S^1)\cong\mu_n\cong\mathbb{Z}/n\mathbb{Z},$$

where μ_n is the group of n'th roots of unity. To See this isomorphism we consider that map

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z},S^1)\to\mu_n \text{ by } \chi\mapsto\chi(1).$$

This map is clearly a well-defined homomorphism, as $\chi(1)^n=\chi(n)=\chi(0)=1$, i.e. $\chi(1)\in\mu_n$. If $\chi(1)=1$ then $\chi(m)=\chi^m(1)=1$, which proves the injectivity. Equality of the cardinalities concludes the proof.

2.7. **Exercise.** Recall the the product expansion

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(1) Use the above formula to prove that,

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left[\frac{1}{z-d} + \frac{1}{z+d}\right] = \pi \cot(\pi z) = \pi i - 2\pi i \sum_{d=0}^{\infty} e(dz).$$

(2) Prove that for even natural number k

$$\zeta(k) = -\frac{(2\pi i)^k}{2k!} B_k,$$

where B_k are the Bernoulli numbers.

(3) Prove that $\zeta(s)$ has zeros at negative even integers.
Hint: Use the functional equation of $\zeta(s)$.

2.8. **Solution.**

(1) We do a logarithmic differentiation of the given expression.

$$\begin{aligned}\pi \cot(\pi z) &= \frac{d}{dz} \log \sin(\pi z) \\ &= \frac{d}{dz} \log(\pi z) + \frac{d}{dz} \sum_{n=1}^{\infty} \log(1 - z^2/n^2) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2},\end{aligned}$$

hence the first equality. For the second equality we see that,

$$\pi \cot(\pi z) = \pi i \frac{e(z)+1}{e(z)-1} = \pi i - 2\pi i \frac{1}{1-e(z)} = \pi i - 2\pi i \sum_{n=0}^{\infty} e(nz),$$

completing the proof.

(2) Recall that the Bernoulli numbers are defined by the coefficient of the series expansion of $\frac{x}{e^x - 1}$, i.e.

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.$$

Consider the generating series of $\zeta(2k)$

$$1 + 2\sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

For $|z|< 1$ the above sum is absolutely convergent, so plugging in the definition of $\zeta(s)$ for $s > 1$ and changing the order of the summation we get that above sum is

$$1 + 2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(z/n)^{2k} = 1 + \sum_{n=1}^{\infty}\frac{2z^2}{n^2 - z^2} = \pi z\cot(\pi z),$$

where the last equality is from (1). But from (2)

$$\pi z\cot(\pi z) = \pi iz - \frac{2\pi iz}{1 - e^{2\pi iz}} = \pi iz - \sum_{k=0}^{\infty}B_k\frac{(2\pi iz)^k}{k!}.$$

Equating two power series we conclude that

$$2\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!},$$

concluding the result.

(3) We recall the functional equation of $\zeta(s)$

$$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \zeta(1-s)\pi^{(1-s)/2}\Gamma((1-s)/2).$$

We also recall the duplication formula,

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma(s/2)\Gamma((1+s)/2),$$

and

$$\Gamma(1/2 - s/2)\Gamma(1/2 + s/2) = \frac{\pi}{\cos(\pi s/2)}.$$

Combining all of them we get that,

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s).$$

Plugging in $s = 2n+1$ for $n \geq 1$ and checking that $\cos(n\pi + \pi/2) = 0$ we conclude that

$$\zeta(-2n) = 0.$$

2.9. **Eisenstein Series of weight 2.** In the lecture we have defined Eisenstein series E_k of weight k for $k > 2$. In this exercise we will define *Eisenstein series* E_2 *of weight* 2 and will show that it satisfies an "almost modularity" relation.

2.10. **Exercise.** Define the following functions for $z \in \mathbb{H}$:

$$G_2(z) := \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2},$$

$$G_2^*(z) := G_2(z) - \frac{\pi}{2\Im(z)},$$

$$G_{2,\epsilon} := \frac{1}{2}\sum_{(m,n)\neq(0,0)}\frac{1}{(mz+n)^2}\frac{1}{|mz+n|^{2\epsilon}}, \text{ for } \epsilon > 0.$$

(1) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove that $G_{2,\epsilon}$ converges absolutely and locally uniformly. Also show that,
$$G_{2,\epsilon}(\gamma z) = (cz+d)^2|cz+d|^{2\epsilon}G_{2,\epsilon}(z).$$
(2) For $\epsilon > -1/2$ define:
$$I_\epsilon(z) := \int_{\mathbb{R}} \frac{dt}{(z+t)^2|z+t|^{2\epsilon}} \text{ and } I(\epsilon) := \int_{\mathbb{R}} \frac{dt}{(i+t)^2(1+t^2)^\epsilon}.$$
Consider
$$G_{2,\epsilon}(z) - \sum_{m=1}^{\infty} I_\epsilon(mz).$$
Use the mean value theorem to prove that it converges absolutely and locally uniformly for $\epsilon > -1/2$ and the limit as $\epsilon \to 0$ is $G_2(z)$.
(3) Show that
$$I_\epsilon(z) = \frac{I(\epsilon)}{\Im(z)^{1+2\epsilon}} \text{ and } I'(0) = -\pi.$$
Use this to show that the limit of $G_{2,\epsilon}(z)$ as $\epsilon \to 0$ is $G_2^*(z)$. Hence G_2^* transforms like a modular form of weight 2.
(4) Conclude that
$$G_2(\gamma z) = (cz+d)^2 G_2(z) - \pi i c(cz+d).$$
E_2 is defined to be, as usual, $\frac{G_2}{\zeta(2)}$.

2.11. **Solution.**

(1) Note that, for $k > 2$ and $z \in \mathbb{H}$
$$\sum_{N=1}^{\infty} \sum_{N<|mz+n|\leq N+1} \frac{1}{|mz+n|^k} \leq \sum_{N=1}^{\infty} \frac{\#\{(m,n) \in \mathbb{Z}^2 \mid N \leq |mz+n| \leq N+1\}}{N^k}.$$
It is easy to check that
$$\#\{(m,n) \mid N \leq |mz+n| \leq N+1\} \ll \pi(N+1)^2 - \pi N^2 \ll N.$$
Thus the above sum is, as $k > 2$
$$\ll \sum_{N=1}^{\infty} N^{1-k} < \infty.$$
Now we see that,
$$G_{2,\epsilon} \leq \sum_{0\leq|mz+n|\leq 1} |mz+n|^{-2-2\epsilon} + \sum_{1\leq|mz+n|} |mz+n|^{-2-2\epsilon}.$$
The first sum has finite number of summands and second sum is absolutely and locally uniformly convergent by the previous argument. Thus the sum of $G_{2,\epsilon}$ are

convergent abolustely and locally uniformly, thus defines a holomorphic function on $\mathbb{H}$. To see the transformation law we first note that every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ induces a bijection from $\mathbb{Z}^2 \setminus \{(0,0)\}$ to itself by right multiplication. Also one checks that,

$$m\gamma z + n = \frac{(ma+nc)z + (mb+nd)}{cz+d} = \frac{m'z+n'}{cz+d}.$$

Combining these two facts, we conclude that

$$G_2,\epsilon(\gamma z) = \sum_{(m',n')\neq(0,0)} \frac{(cz+d)^2|cz+d|^{2\epsilon}}{(m'z+n')|m'z+n'|^{2\epsilon}} = (cz+d)^2|cz+d|^{2\epsilon}G_{2,\epsilon}(z).$$

(2) Let

$$f(t) := (mz+t)^2|mz+t|^{-2\epsilon},$$

with implicit dependence on mz. Now as we have proved the absolute convergence of the $\sum f(n)$ we will freely change the order of summations and order of integration and summation, as follows.

$$\begin{aligned}\tilde{G}_{2,\epsilon}(z) &= G_{2,\epsilon}(z) - \sum_{m=0}^{\infty} I_\epsilon(mz) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}(f(n) - \int_n^{n+1} f(t)dt) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\int_n^{n+1}(f(n)-f(t))dt.\end{aligned}$$

By the mean value theorem on $n \leq t \leq n+1$ we get that

$$|f(n)-f(t)| \leq \sup_{n\leq u\leq n+1} |f'(u)| \ll |mz+n|^{-3-2\epsilon}.$$

Hence, the sum is absolutely convergent for $\epsilon > -1/2$ and thus $\lim_{\epsilon\to 0}\tilde{G}_{2,\epsilon}$ exists and defines a holomorphic function. We calculate,

$$\begin{aligned}&\lim_{\epsilon\to 0}\tilde{G}_{2,\epsilon}(z) \\ &= \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \sum_{m=1}^{\infty}\left[\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2} + \sum_{n\in\mathbb{Z}}\left(\frac{1}{mz+n+1} - \frac{1}{mz+n}\right)\right] \\ &= \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2} \\ &= G_2(z)\end{aligned}$$

(3) Let $z = x + iy$. Then changing variable $t \mapsto yt - x$ we get that,

$$I_\epsilon(x+iy) = \int_{\mathbb{R}} \frac{dt}{(x+t+iy)^2|x+t+iy|^{2\epsilon}}$$
$$= \frac{1}{y^{1+2\epsilon}} \int_{\mathbb{R}} \frac{dt}{(t+i)^2|t+i|^{2\epsilon}} = \frac{I(\epsilon)}{y^{1+2\epsilon}}.$$

Differentiating under the integration sign and then integrating by parts we get that,

$$I'(0) = -\int_{\mathbb{R}} \frac{\log(1+t^2)}{(t+i)^2} dt = \frac{\log(1+t^2)}{t+i}\Bigg|_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{2tdt}{(t+i)(1+t^2)}$$
$$= -\int_{\mathbb{R}} \frac{1}{(t+i)^2} + \frac{1}{1+t^2} = -\int_{\mathbb{R}} \frac{dt}{t^2+1} = -\pi.$$

Using the above two results we compute that,

$$\lim_{\epsilon\to 0} \sum_{m=1}^{\infty} I_\epsilon(mz) = \lim_{\epsilon\to 0} \sum_{m=1}^{\infty} \frac{I(\epsilon)}{(my)^{1+2\epsilon}} = \lim_{\epsilon\to 0} \frac{I(\epsilon)\zeta(1+2\epsilon)}{\Im(z)^{1+2\epsilon}}.$$

From the exercise 1.6 we know that

$$\zeta(1+2\epsilon) = \frac{1}{2\epsilon} + O(1).$$

Using that $I(0) = 0$ we have that above limit equals to

$$\lim_{\epsilon\to 0} \frac{I(\epsilon)}{2\epsilon\Im(z)^{1+2\epsilon}} = \frac{I'(0)}{2\Im(z)}.$$

Thus,

$$\lim_{\epsilon\to 0} G_{2,\epsilon}(Z) = \lim_{\epsilon\to 0}\left(\tilde{G}_{2,\epsilon}(z) + \sum_{m=1}^{\infty} I_\epsilon(mz)\right) = G_2(z) - \frac{\pi}{2\Im(z)} = G_2^*(z).$$

(4) From part (1) and (3) letting $\epsilon \to 0$ we see that $G_2^*(z)$ transforms as a modular form of weight 2. So,

$$G_2(\gamma z) - (cz+d)^2 G_2(z) = \frac{\pi}{2\Im(\gamma z)} - (cz+d)^2 \frac{\pi}{2\Im(z)}$$
$$= \frac{\pi}{2\Im(z)}(|cz+d|^2 - (cz+d)^2)$$
$$= \pi i c(cz+d),$$

concluding the result.

3. Due on 24th October

3.1. **Exercise.** Prove the Bruhat decomposition: for any subfield $K \subset \mathbb{C}$

$$\mathrm{SL}_2(K) = N(K)A(K) \sqcup N(K)wN(K)A(K),$$

where the notatons are same as in the lectures. Using this prove that the fractional linear transformation $\mathrm{GL}_2(\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C})$ preserves the lines.

3.2. **Solution.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. if $c = 0$ then g is upper triangual so lies in NA. So let us assume that $c \neq 0$. So $b = ad/c$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1/c \end{pmatrix}.$$

This also can be proved in much more geometric way. First check that

$$g.\infty = a/c \implies \mathrm{Stab}_{\mathrm{GL}(2)}(\infty) = NA.$$

We prove that if $g \notin NA$ then $g \in NwNA$. To check this we see that

$$\begin{pmatrix} 1 & -a/c \\ & 1 \end{pmatrix} g.z = g.z - a/c = \frac{az+b}{cz+d} - \frac{a}{c} = \frac{1}{c^2z+cd} = w.c^2z + cd = w \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1/c \end{pmatrix} .z.$$

To check that this decombosition is unique we note that, again, if $g = b \in NA$ this is obvious. If $g = nwb = n'wb'$ then

$$g.\infty = n.0 = n'.0 \implies n = n' \implies b = b'.$$

This proves the first part.
For the second part we first recall that a line in $\mathbb{P}^1(\mathbb{C})$ is of the form $L \cup \{\infty\}$ where L is a line or a circle in $\mathbb{C}$. As from the previous part and the fact that

$$\mathrm{GL}_2(\mathbb{C}) \cong Z(\mathbb{C})\mathrm{SL}_2(\mathbb{C}),$$

it is enough to prove that Z, N, A, w preserves the lines. While Z, N, A transforms in affine way, i.e.

$$z \mapsto az + b, \quad a \in \mathbb{C}^\times, b \in \mathbb{C}$$

it is clear that they preserve lines. Thus it is enough to check that w preserves a line L. Now, as we can freely move object in affine way, we may assume that L is a horizontal line passing through 0, i.e. $\Im(z) = 0$ or a unit circle centered at origin, i.e. $|z| = 1$. In either case the fact that

$$w.z = -\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

proves the claim.

3.3. **Exercise.** Recall the Fourier expansions of the Eisenstein series

$$E_k(z) = 1 + c_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where for $k = 2, 4, \ldots, 14$ the c_k are $-24, 240, -504, 480, -264, 65520/691, -24$ with $q := e(z)$ and $\sigma_s(n) := \sum_{d|n} d^s$.

(1) Use dimension formula to show that $E_8 = E_4^2$, $E_4E_6 = E_{10}$, and $E_6E_8 = E_{14}$. What relations can you get between σ_n's using the above relations (some of them were obtained during the lectures)?
(2) Define the *Serre derivative* by

$$D_k := \frac{1}{2\pi i}\frac{d}{dz} - \frac{k}{12}E_2.$$

Show that $D_k : M_k \to M_{k+2}$ and $D_k f \in S_{k+2}$ iff $f \in S_k$.
(3) Calculate DE_4 and DE_6. Find σ_5 in terms of σ_1 and σ_3 resp. and σ_7 in terms of σ_1 and σ_5.

3.4. **Solution.**

(1) Check that from the dimension formula that m_8, M_10, and M_14 are one dimensional. Therefore, $E^8 - cE_4^2$, $E_4E_6 = de_{10}$, and $E_6E_8 = eE_{14}$. But from the Fourier expansions of the Eisenstein series that their first Fourier coefficients are one we cocnlude that $c = d = e = 1$. Now multiplying the Fourier expansions we get that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m),$$

$$-11\sigma_9(n) = 10\sigma_3(n) - 21\sigma_5(n) - 5040\sum_{m=1}^{n-1}\sigma_3(m)\sigma_5(n-m),$$

$$-\sigma_{13}(n) = -21\sigma_5(n) + 20\sigma_7(n) - 10080\sum_{m=1}^{n-1}\sigma_5(m)\sigma_7(n-m).$$

(2) Let $f \in M_k$. As E_2, f, and f' are holomorphic so is $D_k f$. So it is enough to show that $D_k f$ transforms as a weight $k+2$ form to prove that image of D_k is in M_{k+2}. We check that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $j(\gamma, z) = cz + d$, and recalling from exercise 2.10(4) that

$$E_2(\gamma z) = j(\gamma, z)^2 E_2(z) + \frac{12cj(\gamma, z)}{2\pi i}.$$

we get that

$$\begin{aligned}
D_k f(\gamma z) &= \frac{1}{2\pi i} f'(\gamma z) - \frac{k}{12} E_2(\gamma z) f(\gamma z) \\
&= \frac{1}{2\pi i} j^2(\gamma, z) \frac{df(\gamma z)}{dz} - j^{k+2}(\gamma, z) E_2(z) f(z) - \frac{ckj^{k+1}(\gamma, z)}{2\pi i} f(z) \\
&= \frac{1}{2\pi i} j^2(\gamma, z) \frac{d}{dz} j^k(\gamma, z) f(z) - j^{k+2}(\gamma, z) E_2(z) f(z) - f(z) \frac{j^2(\gamma, z)}{2\pi i} \frac{d}{dz} j^k(\gamma, z) \\
&= \frac{j^{k+2}(\gamma, z)}{2\pi i} f'(z) - j^{k+2}(\gamma, z) E_2(z) f(z) \\
&= j^{k+2}(\gamma, z) D_k f(z).
\end{aligned}$$

Now note that,

$$q = e(z) \implies \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}.$$

Thus if f has Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

then

$$D_k f = q\frac{df}{dq} - \frac{k}{12} E_2 f = \sum_{n=0}^{\infty} n a_n q^n + \frac{k}{12} E_2 f.$$

Thus it is clear that the zeroth Fourier coefficient is $-ka_0/12$ and that will be zero if and only if $a_0 = 0$ which proves the second claim.

(3) By part (2) $DE_4 \in M_6$ and $DE_6 \in M_8$. From the dimension formulas and the zeroth Fourier coefficients we conclude as in (1) that

$$DE_4 = cE_6, \quad c \in \mathbb{C},$$

with $c = -1/3$. Similarly, $DE_6 = -\frac{1}{2}E_8$. Now as in (1) comparing the Fourier coefficients we get that

$$21\sigma_5(n) = (30n - 10)\sigma_3(n) + \sigma_1(n) + 240 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_3(n-m),$$

$$20\sigma_7(n) = (42n - 21)\sigma_5(n) + \sigma_1(n) + 504 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_5(n-m).$$

3.5. **Exercise.** Recall that the Delta function from the lecture defined in terms of some Eisenstein series. Here we start with a different defintion and show equality afterwards.

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

which has a Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty}\tau(n)q^n = q - 24q^2 + 252q^3 + O(q^4) \in \mathbb{Z}[[q]],$$

with $q = e(z)$ as usual. $\tau : \mathbb{N} \to \mathbb{C}$ is called *Ramanujan Tau function.*

(1) Prove that $\frac{1}{2\pi i}\frac{d}{dz}\log\Delta(z) = E_2(z)$ and conclude that $\Delta \in S_{12}$.
(2) Show that $\Delta = \frac{E_4^3 - E_6^2}{1728}$, and derive τ in terms of σ_3 and σ_5.
(3) Show that $E_{12} - E_6^2 = c\Delta$ with $c = \frac{2^6 3^5 7^2}{691}$ and derive relation between τ, σ_{11} and σ_5. Use this to prove the famous congruence by Ramanujan:

$$\tau(n) \equiv \sigma_{11}(n) \mod 691,$$

for all $n \geq 1$.

3.6. **Solution.**

(1) Recall that $\frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$. Therefore,

$$\begin{aligned}\frac{1}{2\pi i}\frac{d}{dz}\log\Delta(z) &= q\frac{d}{dq}\log\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right)\\ &= q\frac{d}{dq}\left[\log q + 24\sum_{n=1}^{\infty}24\log(1-q^n)\right]\\ &= q\frac{d}{dq}\left[\log q - 24\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{q^{nk}}{k}\right]\\ &= 1 - 24\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}nq^{nk}\\ &= 1 - 24\sum_{n=1}^{\infty}q^n\left(\sum_{k|n}k\right) = E_2(z).\end{aligned}$$

All interchanges of orders of summations are justified as the series is absolutely convergent as $|q|< 1$. Now from the product form it is clear that Δ is holomorphic and has zero as zeroth Fourier coefficient. So to prove that $\Delta \in S_{12}$ it is enough to show that Δ transforms as a weight 12 modular form. To check that keeping the same notations as in the solution 3.4(2) we compute that

$$\begin{aligned}
&\frac{1}{2\pi i}\frac{d}{dz}\log\Delta(\gamma z)\\
&= j(\gamma z)^{-2}\frac{1}{2\pi i}\frac{d}{dz}\log\Delta|_{\gamma z}\\
&= j(\gamma z)^{-2}E_2(\gamma z)\\
&= E_2(z)+\frac{12c}{2\pi i j(\gamma,z)}\\
&= \frac{1}{2\pi i}\frac{d}{dz}\log\Delta(z)+\frac{1}{2\pi i}\frac{d}{dz}\log j^{12}(\gamma,z)\\
&= \frac{1}{2\pi i}\frac{d}{dz}\log(j^{12}(\gamma,z)\Delta(z)).
\end{aligned}$$

Thus for each $\gamma\in\mathrm{SL}_2(\mathbb{Z})$ there exists a constant $0\neq c(\gamma)$ such that

$$\Delta(\gamma z)=c(\gamma)j^{12}(\gamma,z)\Delta(z).$$

It suffices to show that $c(\gamma)=1$ for all γ. It is easy to check that

$$c:\mathrm{SL}_2(\mathbb{Z})\to\mathbb{C}^\times,\quad \gamma\mapsto c(\gamma)$$

a character. Thus it is enough to prove that $c(T)=1$ and $c(S)=1$ where T,S are the usual generators of $\mathrm{SL}_2(\mathbb{Z})$. But as Δ is 1-periodic so $c(T)=1$. Now as $S.i=i$ and $\Delta(i)\neq 0$ we see that

$$c(S)=i^{-12}=1,$$

completing the proof.

(2) As S_{12} is one dimensional and $E_4^3-E_6^2$ has zero zeroth Fourier coefficient hence,

$$E_4^3-E_6^2=d\Delta,\quad d\in\mathbb{C}.$$

d can be calculated to be 1728 from the first Fourier coefficients of E_4 and E_6. Thus equating Fourier coefficients we conclude that

$$\begin{aligned}
12\tau(n)&=5\sigma_3(n)+1200\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m)+96000\sum_{r=1}^{n-1}\sum_{m=1}^{r-1}\sigma_3(m)\sigma_3(r-m)\sigma_3(n-r)\\
&\quad+7\sigma_5(n)-1764\sum_{m=1}^{n-1}\sigma_5(m)\sigma_5(n-m).
\end{aligned}$$

(3) Again by dimension formula arguing that S_{12} is one dimensional and comparing the first Fourier coefficients we conclude that

$$E_{12}-E_6^2=\frac{2^6 3^5 7^2}{691}\Delta.$$

Comparing the Fourier coefficients we get that

$$2^6 3^5 7^2 \tau(n) = 65520\sigma_{11}(n) + 691.2.504\sigma_5(n) - 691.504^2 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).$$

Dividing by 1008 and reducing mod 691 we conclude that

$$756\tau(n) \equiv 65\tau(n) \equiv 65\sigma_{11}(n) \mod 691.$$

As $(65, 691) = 1$ we conclude the final result.

3.7. **A Riemmanian metric on the upper half plane.** A Riemmanian metric on $\mathbb{H}$ can be defined as

$$ds^2(z) = \frac{d\Re^2(z) + d\Im^2(z)}{\Im^2(z)},$$

which gives $\mathbb{H}$ a hyperbolic structure (More details in the upcoming lecture).

3.8. **Exercise.** Let $z_1, z_2 \in \mathbb{H}$. We define *geodesic segment between* z_1 *and* z_2 to be the unique length minimizing curve (which exists) joining z_1 and z_2 under the hyperbolic metric as above. We define the hyperbolic distance between z_1 and z_2 to be

$$d_h(z_1, z_2) := \text{Length of geodesic segment between } z_1 \text{ and } z_2.$$

(1) Prove that
$$ds^2(gz) = ds^2(z), \quad \forall g \in \mathrm{GL}_2^+(\mathbb{R}),$$
that is ds^2 is a $\mathrm{GL}_2^+(\mathbb{R})$ invariant metric.
(2) Prove that if $\Re(z_1) = \Re(z_2)$ then the geodesic segment joining them is the vertical line joining z_1 and z_2.
(3) Prove that for general z_1 and z_2 the geodesic segment joining them is the arc of the unique half-circle centered on $\mathbb{R}$ containing these two points.
(4) Prove that
$$\cosh(d_h(z_1, z_2)) = 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)}.$$

3.9. **Solution.**

(1) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we check that
$$\frac{d(gz)}{dz} = \frac{\det(g)}{(cz+d)^2}.$$
Also recall that
$$\Im(gz) = \frac{\det(g)\Im(z)}{|cz+d|^2}.$$
Thus
$$ds^2(gz) = \frac{|d(gz)|^2}{\Im(gz)^2} = \frac{|\det(g)|^2}{|cz+d|^4}|dz|^2 \frac{|cz+d|^4}{|\det(g)|^2\Im(z)^2} = \frac{|dz|^2}{\Im(z)^2} = ds^2(z).$$

(2) WLOG let $\Im(z_2) \geq \Im(z_1)$. Note that, the vertical path joining z_1 and z_2 can be given as

$$\phi(t) = \Re(z_1) + i\Im(z_1)\left(\frac{\Im(z_2)}{\Im(z_1)}\right)^t.$$

It is easy to check that the length of ϕ

$$L(\phi) = \log \Im(z_2) - \log \Im(z_1).$$

Let ϕ' be any other curve joining z_1 and z_2. Then the length of ϕ_1

$$L(\phi_1) = \int_0^1 \frac{|\phi_1'(t)|}{\Im(\phi_1(t))}dt \geq \int_0^1 \frac{\Im(\phi_1'(t))}{\Im(\phi_1(t))}dt = \log \Im(z_2) - \log \Im(z_1),$$

which proves the claim.

(3) First we claim that there exists a $g \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\Re(gz_1) = \Re(gz_2) = 0.$$

First we assume the claim. Then we see that the length minimizing curve joining gz_1 and gz_2, them having same real part, is a vertical segment ϕ as in the previous part. As $\mathrm{SL}_2(\mathbb{R})$ acts by isometry the geodesic joining z_1 and z_2 would be $g^{-1}\phi$. From Exercise 3.1 we can conclude that $\mathrm{SL}_2(\mathbb{R})$ preserves lines in $\mathbb{P}^1(\mathbb{R}) \cong \mathbb{H} \cup \{\infty\}$, where lines in $\mathbb{P}^1(\mathbb{R})$ are vertical lines or half-circles centered in $\mathbb{R}$. This concludes the proof assuming the claim.
Now we turn to prove the claim. By transitivity property of $\mathrm{SL}_2(\mathbb{R})$ action one can find g such that $gz_1 = i$. Now as we know that $\mathrm{SO}(2)$ fixes i for any $k \in \mathrm{SO}(2)$ we have $gki = z_1$. So it is enough to find some k such that $\Re(kg^{-1}z_2) = 0$. For any $z \in \mathbb{H}$ we can always find $k \in \mathrm{SO}(2)$ such that $\Re(kz) = 0$. If $k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $z = x + iy$) then to make sure that $\Re(kz) = 0$ one needs to see whether

$$\tan(2\theta) = -\frac{x}{y^2 + 1 - x^2},$$

which clearly exists.

(4) By the argument in the part (3) we can find $g \in \mathrm{SL}_2(\mathbb{R})$ such that gz_1 and gz_2 has zero real parts. Also from part (1) we know that g acts by isometry thus it is enough to prove the statement for z_1 and z_2 purely imaginary. But in part (2) we have proved that for such $z_i \in i\mathbb{R}$ one has

$$d_h(z_1, z_2) = |\log \Im(z_1) - \log \Im(z_2)| = |\log(z_1/z_2)|.$$

Thus,

$$\begin{aligned}\cosh(d_h(z_1,z_2)) &= \frac{1}{2}\left(e^{d_h(z_1,z_2)} + e^{-d_h(z_1,z_2)}\right) \\ &= \frac{1}{2}\left|\frac{z_1}{z_2} + \frac{z_1}{z_2}\right| 2 = \frac{|z_1^2 + z_2^2|}{2|z_1 z_2|} \\ &= 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)},\end{aligned}$$

completing the proof.

4. Due on 7th November

4.1. **Exercise.** Prove that all the geodesics of $\mathbb{H}$ are the perpendicular lines $\mathbb{P}^1(\mathbb{R})$ at two points.

4.2. **Solution.** From the Exercise 3.8(2), 3.8(3) we know the the geodesic joining two points z_1 and z_2 in $\mathbb{H}$ is the arc of the unique half-circle centered on $\mathbb{R}$ containing these two points if they have different real parts and the vertical lines joining them if they have same real parts. So we need to check that both the semicircles centered in $\mathbb{R}$ and the vertical lines are perpendicular to $\mathbb{P}^1(\mathbb{R})$ at two points. As $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, semicircles are clearly perpendicular to $\mathbb{R}$ at two points in $\mathbb{R}$, where as, vertical lines are perpendicular at a point in $\mathbb{R}$ and ∞. This proves the claim.

4.3. **Exercise.** Recall the canonical projection map

$$\pi : \mathbb{H} \to Y(\Gamma) := \{\Gamma z \mid z \in \mathbb{H}\}.$$

Let $U_i \subset \mathbb{H}$ be an open set. Prove that

(1) $\pi(U_1) \cap \pi(U_2) = \varnothing$ in $Y(\Gamma)$ iff $\Gamma(U_1) \cap U_2 = \varnothing$ in $\mathbb{H}$.
(2) $Y(\Gamma)$ is connected.

4.4. **Solution.**

(1) We will show the contrapositive, that

$$\pi(U_1) \cap \pi(U_2) \neq \varnothing \iff \Gamma(U_1) \cap U_2 \neq \varnothing.$$

To see this let for some $u_i \in U_i$ for $i = 1, 2$

$$\Gamma u_1 = \Gamma u_2.$$

This implies that for $\gamma_1 \in \Gamma$ there exists $\gamma_2 \in \Gamma$ such that

$$\gamma_1 u_1 = \gamma_2 u_2 \implies \gamma_2^{-1} \gamma_1 u_1 = u_2.$$

But the above implies that $u_2 \in \Gamma u_1$, in other words,

$$u_2 \in \Gamma(U_1) \cap U_2.$$

The opposite implication is trivial. If there exists $u_i \in U_i$ for $i = 1, 2$ such that $u_2 \in \Gamma u_1$ then $\Gamma u_1 = \Gamma u_2$. Hence $\pi(U_1) \cap \pi(U_2) \neq \varnothing$.

(2) As π is a projection, hence a continuous surjection, and $\mathbb{H}$ is connected so $\mathrm{Im}(\pi) = Y(\Gamma)$ is also connected.

4.5. **Exercise.** Recall the definition of $U_{x,Y}$ from the lecture with $x \in \mathbb{P}^1(\mathbb{Q})$ and $Y > 0$. Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, $x, y \in \mathbb{P}^1(\mathbb{Q})$, and $z \in \mathbb{H}$.

(1) Let U be a neighbourhood of z with compact closure. Show that the set
$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U \neq \varnothing\}$$
is finite, in fact empty, for U sufficient small and Y sufficiently large.

(2) If $y \notin \Gamma x$ then show that the set
$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U_{y,Y} \neq \varnothing\}$$
is finite for any $Y > 0$ and empty if $Y > 1$.

(3) If $Y > 1$ then show that
$$\{\gamma \in \Gamma \mid \gamma U_{x,Y} \cap U_{x,Y} \neq \varnothing\} = \Gamma_x.$$

(4) Prove that $X(\Gamma)$ equipped with the quotient topology is a connected, compact, Hausdorff topological space.

4.6. **Solution.**

(1) Note that
$$\gamma U_{x,Y} = \gamma\sigma_x(H_Y \cup \{\infty\}) = \gamma\sigma_x H_Y \cup \gamma\{x\}.$$
As $x \notin \mathbb{H}$ and U has compact closure hence
$$u \in \gamma U_{x,Y} \cap U \implies u \in \gamma\sigma_x H_Y.$$
Now as σ_x is determined up to a translation on the right and H_Y is translation invariant we may think that $x = \infty$ and count $\gamma \in \Gamma$ so that $\gamma u \in H_Y$ with $u \in U$. In the usual notation of γ this implies that
$$\frac{\Im(u)}{|cu+d|^2} > Y \implies \min\left\{\frac{1}{c^2\Im(u)}, \frac{\Im(u)}{(c\Re(u)+d)^2}\right\} > Y.$$
As U has compact closure both c ad d has finitely many choices, following a similar argument as in the lecture. Thus there are only finitely many (c, d) such that γ with bottom row (c, d) has $\gamma U_{x,Y} \cap U \neq \varnothing$. This in turn, equivalently, implies that there are finitely many $\gamma \in \Gamma_x \backslash \Gamma$ such that the same happens. In fact, if
$$Y > \sup_{u \in U}\{\Im(u), \Im(u)^{-1}\},$$
then from the above inequalities we conclude that $c = 0 = d$, thus no possible choice for γ.

(2) By conjugating we may assume that $y = \infty$. So, as $U_{x,Y} = \sigma_x H_Y \cup \{x\}$ and $\infty \notin \Gamma x$, we have that
$$u \in \gamma U_{x,Y} \cap U_{\infty,Y} \implies \gamma^{-1}\sigma_x^{-1}u, u \in H_Y.$$
We count $\gamma' := \gamma^{-1}\sigma_x^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Again proceeding as previous we see that c has finitely many choice and thus finitely many choices for γ. This proves the claim.

(3) As we know that $\sigma_x^{-1}x = \infty$ and $\sigma_x^{-1}\Gamma_x\sigma_x = B$ we may conjugate the claimed equation by σ_x and assume that $x = \infty$. Therefore, if $u \in \gamma^{-1}U_{x,Y} \cap U_{x,Y}$ we have that $\Im(\gamma u) > Y$ and $\Im(u) > Y$. This implies that

$$Y > \Im(\gamma u) = \frac{\Im(u)}{|cu+d|^2} \geq \frac{1}{c^2\Im(u)} < c^{-2}Y^{-1}.$$

As $Y > 1$ this implies that $c = 0$ and thus $\gamma \in B$. Other inclusion is trivial to show.

(4) Compactness and connectedness of $X(\Gamma)$ follow from compactification and connectedness of compactification of connected space respectively. Hausdorff property follows from part (1) and (2).

4.7. **Exercise.** Show that a set of representatives for $\mathrm{Cusp}(\Gamma_0(q))$ is given by the fractions

$$\left\{\frac{u}{v}\middle| v \mid q, 0 < u \leq (v, q/v)\right\}.$$

Compute their respective widths.

4.8. **Solution.** All cusps are equivalent to some rational numbers as $\Gamma_0(q) \subset \mathrm{SL}_2(\mathbb{Z})$. First we find a set of representatives of $\Gamma_0(q)\backslash\mathrm{SL}_2(\mathbb{Z})$. We claim that they are given by

$$\begin{pmatrix} a & * \\ c & * \end{pmatrix}, \quad \text{with } c \mid q, (a,c) = 1 \text{ and } a \mod q/c.$$

To see this we check that

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ c'a + d'c & c'b + d'd \end{pmatrix}.$$

Here $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(q)$. So$(a,q)$ is invariant by multiplication of $\Gamma_0(q)$ in the left. In fact, we can choose c', d' such that

$$v := c'b + d'd = (d,q) \implies v \mid q.$$

Solutions of $c'b + d'd = v$ form an one parameter family $(c' + dt, d' - bt)$ where t ranges over mod q/v, to ensure $c' \equiv 0 \mod q$. These solutions translate the bottom left $u := c'a + d'c$by t, which ensures any choice of u modulo q/v. So a set of representatives can be chosen as

$$\tau = \begin{pmatrix} * & * \\ a & c \end{pmatrix}, \quad \text{with } c \mid q, (a,c) = 1 \text{ and } a \mod q/c.$$

Now transforming

$$\tau \mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\tau^{-1}$$

we conclude the claim.

Now the cusps of $\Gamma_0(q)$ are $\Gamma_0(q)\backslash SL_2(\mathbb{Z}).\infty$. They are of the form

$$\{a/c \mid \text{with } c \mid q, (a,c) = 1 \text{ and } a \mod (c, q/c)\}.$$

To see that they are non-equivalent, let

$$\frac{a'}{c'} = \gamma.\frac{a}{c},$$

for some $\gamma = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma_0(q)$. So $c' = c_1 a + d_1 c$. This implies that $c \mid c'$ and as $(c', d_1) = 1$ also $c' \mid c$ so $c = c'$. Thus $d_1 \equiv 1 \mod q/c$. So

$$a' = a_1 a + b_1 c \equiv a_1 a \equiv d_1 a \equiv a \mod (c, q/c).$$

This proves the set of inequivalent cusps is given by the claimed formula.

To find the width of the cusp $\mathfrak{a} := a/c$ with need to find the generator of $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} \subset B$. Let the generator be $\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$. Thus

$$\begin{aligned} \Gamma_{\mathfrak{a}} &= \sigma_{\mathfrak{a}}\Gamma_\infty\sigma_{\mathfrak{a}}^{-1} \cap \Gamma_0(q) \\ &= \left\{ \pm \begin{pmatrix} a & * \\ c & * \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \begin{pmatrix} * & * \\ -c & a \end{pmatrix} \in \Gamma_0(q) \right\} \\ &= \left\{ \pm \begin{pmatrix} 1 - mac & ma^2 \\ mc^2 & 1 + mac \end{pmatrix} : q \mid mc^2 \right\}. \end{aligned}$$

So m ranges over all the multiples of $q/(q, c^2)$ and this shows that $m = q/(q, c^2)$. Thus the width of the cusp a/c is $q/(q, c^2)$.

4.9. **Exercise.** Let $\mathfrak{a}$ and $\mathfrak{b}$ are two cusps for a congruence subgroup Γ with scaling matrices $\sigma_{\mathfrak{a}}$ and $\sigma_{\mathfrak{b}}$.

(1) Prove the following disjoint decomposition of double cosets:

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \delta_{\mathfrak{a}\mathfrak{b}} B \cup \bigcup_{c>0} \bigcup_{d \mod c} B \begin{pmatrix} * & * \\ c & d \end{pmatrix} B,$$

where B is the set of upper triangular matrices in $\mathrm{SL}_2(\mathbb{Z})$ and $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ such that it belongs to $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$.

(2) Define

$$C(\mathfrak{a}, \mathfrak{b}) := \left\{ c > 0 \mid \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\}.$$

Also define $c(\mathfrak{a}, \mathfrak{b})$ to be the smallest element of $C(\mathfrak{a}, \mathfrak{b})$. Let

$$C := \max\{c(\mathfrak{a}, \mathfrak{a}), c(\mathfrak{b}, \mathfrak{b})\}.$$

Prove that for any $X > 0$

$$\sum_{0<c\le X} c^{-1} \left| \left\{ d \mod c \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\} \right| \le C^{-1}X.$$

Hence for any $c \in C(\mathfrak{a}, \mathfrak{b})$ we have

$$\left|\left\{d \mod c \mid \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\}\right| \leq C^{-1}c^2.$$

(3) Let $z \in \mathbb{H}$ and $Y > 0$. Then prove that

$$|\{\gamma \in \Gamma_{\mathfrak{a}}\backslash\Gamma \mid \Im\sigma_{\mathfrak{a}}^{-1}\gamma z > Y\}| - 1 \ll \frac{1}{c(\mathfrak{a}, \mathfrak{a})Y}.$$

4.10. **Solution.**

(1) First note that for any cusp $\mathfrak{a}$ we have $\sigma_{\mathfrak{a}}^{-1}\mathfrak{a} = \infty$ and $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = B$. We examine the set

$$\Omega_\infty := \{\omega \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \mid \omega\infty = \infty\},$$

which consists of the upper-triangular matrices in $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. Suppose that Ω_∞ is not empty, say $\omega := \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} \in \Omega_\infty$. Evaluating γ at $\mathfrak{b}$, we get that

$$\gamma\mathfrak{b} = \sigma_{\mathfrak{a}}\Gamma\sigma_{\mathfrak{b}}^{-1} = \sigma_{\mathfrak{a}}\infty = \mathfrak{a}.$$

Hence $\mathfrak{a}$ and $\mathfrak{b}$ are equivalent; hence they are same cusps and $\gamma \in \Gamma_{\mathfrak{a}}$ and $\omega \in B$. Therefore, $\Omega_\infty = B$ if $\mathfrak{a} = \mathfrak{b}$, and empty otherwise.

Let $\omega := \begin{pmatrix} a & * \\ c & d \end{pmatrix}$ be any other element of $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. Then from the relation that

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}\begin{pmatrix} a & * \\ c & d \end{pmatrix}\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} = \begin{pmatrix} a + cm & * \\ c & d + cn \end{pmatrix}$$

we can conclude that the double coset $\Omega := B\begin{pmatrix} * & * \\ c & d \end{pmatrix}B$ determines c uniquely and d modulo a multiple of c. Moreover, given c, d with ω, the double coset Ω does not depend on the upper row of ω. To see that if $\omega := \begin{pmatrix} a' & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ then

$$\omega'\omega^{-1} \in \sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = B.$$

Thus $w' \in \omega B$ and so $a' = a + cm$ for some $m \in \mathbb{Z}$. Hence the disjoint decomposition follows.

(2) Let $C = c(\mathfrak{a}) \geq c(\mathfrak{b})$, if not, by symmetry we can interchange the cusps by inversion. If $\omega = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ and $\omega' = \begin{pmatrix} * & * \\ c' & d' \end{pmatrix}$ with $0 < c, c' \leq X$ then $\omega'' := \omega'\omega^{-1} = \begin{pmatrix} * & * \\ c'' & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}}$ with $c'' = c'd - cd'$. If $c'' = 0$ then, as in the previous case, the cusps will be equal. So we assume that $c'' \neq 0$ and thus $|c''| \geq C$ and so

$$\left|\frac{d'}{c'} - \frac{d}{c}\right| \geq \frac{C}{cc'} \geq \frac{C}{cX}.$$

As $(c,d)=1$ the fraction $\frac{d}{c}$ uniquely determines the pair. Let A be set of $\frac{d}{c}$ in $[0,1]$ with the prescribed gap. Then

$$\begin{aligned}1 &\geq \max(A)-\min(A)\\ &\geq \sum_{0<c\leq x}\ \sum_{d/c \text{ and } d'/c' \text{ are successive}} \frac{d}{c}-\frac{d'}{c'}\\ &\geq \frac{C}{X}\sum_{0<c\leq X} c^{-1}\left|\left\{d \mod c \mid \begin{pmatrix} * & * \\ c & d\end{pmatrix}\in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}\right\}\right|,\end{aligned}$$

which completes the proof. Second part is immediate from the proof.

(3) Again as in solution 4.6 we may assume that, possibly conjugating, $\mathfrak{a}=\infty$. Let $\gamma\in\Gamma\setminus\Gamma_{\mathfrak{a}}$, so $c>0$. Also acting by element from Γ_∞ on the left of γz, which amounts to translation of γz we can also assume that

$$z\in\Gamma_\infty\backslash\mathbb{H}\cap D \implies 0<\Re(z)<1, |cz+d|>1.$$

Thus from

$$\Im(\gamma z)=\frac{\Im(z)}{|cz+d|^2}>Y$$

we conclude that $\Im(z)>Y$. Also,

$$c<(\Im(z)Y)^{-1/2}, |c\Re(z)+d|<(\Im(z)/Y)^{1/2}.$$

Thus for $C\leq c<2C$,

$$|\Re(z)+d/c|<\frac{1}{C}(\Im(z)/Y)^{1/2} \implies d/c\in[-1-\frac{1}{C}(\Im(z)/Y)^{1/2},\frac{1}{C}(\Im(z)/Y)^{1/2}].$$

From the spacing property of the possible d/c, as in the previous solution, we conclude that possible number of (c,d) is

$$\ll \frac{C}{c(\infty,\infty)}(\Im(z)/Y)^{1/2}.$$

Now summing over the dyadic intervals with $C=2^{-n}(\Im(z)Y)^{-1/2}$ we get that number of possible γ which are not in B is

$$\ll \sum_{n=1}^{\infty} 2^{-n}\frac{(\Im(z)Y)^{-1/2}}{c(\infty,\infty)}(\Im(z)/Y)^{1/2} \ll \frac{1}{c(\infty,\infty)Y}.$$

Now adding one more point for $\gamma\in B$ we conclude.

5. Due on 21st November

In all the following discussions we let $k > 2$.

5.1. **Poincare series for general cusps.** Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cusps of a congruence subgroup Γ with usual notations for scaling matrices and stabilizing subgroups. Here we will define a *Poincare series of weight k with respect to a cusp which not necessarily* ∞. Recall that $j(g, z) = cz + d$ where g has (c, d) as its lower row. Let $p : \mathbb{H} \to \mathbb{C}$ be a bounded holomorphic function with period one.

(1) Define $\pi : \Gamma \times \mathbb{H} \to \mathbb{C}$ by

$$(\gamma, z) \mapsto j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)^{-k} p(\sigma_{\mathfrak{a}}^{-1}\gamma z).$$

Prove that π is $\Gamma_{\mathfrak{a}}$ left-invariant on the first entry. This allows us to unambiguously define the Poincare series

$$P_{\mathfrak{a}}(z) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \pi(\gamma, z).$$

Check that the defining series of $P_{\mathfrak{a}}$ converges absolutely if $k > 2$.

(2) Prove that $P_{\mathfrak{a}}$ satisfies the modular transformation.

(3) Recall the *slash operation of weight* k, for $\det(g) = 1$, by

$$f_{|g}(z) := j(g, z)^{-k} f(gz).$$

Prove that

$$P_{\mathfrak{a}|\sigma_{\mathfrak{b}}}(z) = \delta_{\mathfrak{a}\mathfrak{b}} p(z) + \sum_{1 \neq \gamma B \backslash \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/B} I_{\gamma}(z),$$

where

$$I_{\gamma}(z) := \sum_{n \in \mathbb{Z}} (c(z+n)+d)^{-k} p\left(\frac{a}{c} - \frac{1}{c(c(z+n)+d)}\right),$$

for any $\gamma = \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$.

(4) In the lecture we have seen the case when $p(z) = e(mz)$ to define m'th Poincare series and obtained its Fourier expansion at the cusp ∞. Prove that, again if $p(z) = e(mz)$ and $\mathfrak{b}$ is a cusp of Γ then the m'th Poincare series $P_{\mathfrak{a}m}$ has Fourier expansion at cusp $\mathfrak{b}$:

$$P_{\mathfrak{a}m}(z) = \sum_{n=1}^{\infty} p_{\mathfrak{a}\mathfrak{b}}(m, n) e(nz),$$

where

$$p_{\mathfrak{a}\mathfrak{b}}(m, n) := (n/m)^{\frac{k-1}{2}} \left\{ \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn} + \frac{1}{(2\pi i)^k} \sum_{c>0} \frac{S_{\mathfrak{a}\mathfrak{b}}(m, n; c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \right\},$$

and

$$S_{\mathfrak{a}\mathfrak{b}}(m,n;c) := \sum_{\begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} e\left(\frac{ma+nd}{c}\right).$$

5.2. **Solution.**

(1) Let $\gamma_{\mathfrak{a}} \in \Gamma_{\mathfrak{a}}$ be any element. As from the definition

$$\gamma' := \sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma^{-1} \in B,$$

thus

$$p(\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\gamma z) = p(\gamma'\sigma_{\mathfrak{a}}^{-1}\gamma z) = p(\gamma'\sigma_{\mathfrak{a}}^{-1}\gamma z),$$

as p is one periodic thus invariant under B. Similarly, j also enjoys such invariance. Hence we conclude the well-definedness of $P_{\mathfrak{a}}$.

To check absolute convergence, we see that, as p is bounded so the series is majorized by

$$\sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} |j(\sigma_{\mathfrak{a}}^{-1}\gamma, z)|^{-k} = \Im(z)^{-k/2} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \Im(\sigma_{\mathfrak{a}}^{1}\gamma z)^{k/2}.$$

One can check that the RHS is absolutely convergent similarly, as in, 2.11(1) and using 4.9(3).

(2) By conjugating the group we assume that $\mathfrak{a} = \infty$ and $\sigma_{\mathfrak{a}} = 1$. Thus

$$P_\infty(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k} p(\gamma z).$$

Note that for some $\tau \in \Gamma$,

$$j(\gamma\tau^{-1}, \tau z) = j(\gamma, z) j(\tau, z)^{-1}.$$

Hence,

$$\begin{aligned} P_\infty(\tau z) &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, \tau z)^{-k} p(\gamma\tau z) \\ &= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma\tau^{-1}, \tau z)^{-k} p(\gamma z) \\ &= j(\tau, z)^k \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} j(\gamma, z)^{-k} p(\gamma z) \\ &= j(\tau, z)^k P_\infty(z). \end{aligned}$$

This shows the modularity of $P_{\mathfrak{a}}$.

(3) First note that,

$$P_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} j(\sigma_{\mathfrak{a}}^{-1}\gamma, \sigma_{\mathfrak{b}}z)^{-k} p(\sigma_{a}^{-1}\gamma\sigma_{\mathfrak{b}}z)$$
$$= j(\sigma_{\mathfrak{b}}, z)^{k} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} j(\sigma_{a}^{-1}\gamma\sigma_{\mathfrak{b}}, z)^{-k} p(\sigma_{a}^{-1}\gamma\sigma_{\mathfrak{b}}z).$$

Now using double coset decomposition in 4.9(1) we conclude the result directly.

(4) This follows exactly same way as shown in the lecture replacing the Bruhat decomposition by general double coset decomposition as in 4.9(1).

5.3. **Petersson trace formula.** If $\mathfrak{a} = \infty$ we will denote $P_{\mathfrak{a}m}$ by P_m. Let f be a weight k modular form having Fourier expansion at $\mathfrak{b}$

$$f(z) = \sum_{n=0}^{\infty} \hat{f}_{\mathfrak{b}}(n)e(nz).$$

Denote $\langle , \rangle$ to be the Petersson inner product as defined in the lecture.

(1) Prove that

$$\langle f, P_m\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\hat{f}_{\infty}(m).$$

(2) Let $\mathcal{F}$ be an orthonormal basis of $S_k(\Gamma)$. Prove that,

$$\frac{\Gamma(k-1)}{(4\pi\sqrt{mn})^{k-1}} \sum_{f\in\mathcal{F}} \hat{f}_{\mathfrak{a}}(m)\overline{\hat{f}_{\mathfrak{b}}(n)} = \delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn} + \frac{1}{(2\pi i)^k}\sum_{c>0}\frac{S_{\mathfrak{a}\mathfrak{b}}(m,n;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right).$$

(3) There exists an absolute constant m_0 such that if $m < m_0 c(\mathfrak{a},\mathfrak{a})$ then $P_{\mathfrak{a}m}$ does not vanish identically.
Hint: Lower bound $\|P_{\mathfrak{a}m}\|^2$ by bounding average Kloosterman sum and the bound $J_k(y) \ll \min(y^k, y^{-1/2})$.

5.4. **Solution.**

(1) Let $z = x + iy$ in usual notation. Using the Fourier expansions of f and p_m at ∞ and doing a folding-unfolding, we calculate

$$\begin{aligned}\langle f, P_m\rangle &= \int_{\Gamma\backslash\mathbb{H}} y^k f(z) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{j(\gamma,z)^{-k} e(m\gamma z)} d\mu z \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}} f(\gamma z)\Im(\gamma z)^k e(-m\gamma z) d\mu z \\ &= \int_0^\infty \int_0^1 f(z) y^{k-2} e(-mz) dx dy \\ &= \sum_{n=0}^{\infty} \hat{f}_\infty(n) \int_0^\infty y^{k-1} e^{-2\pi n y} \frac{dy}{y} \int_0^1 e((n-m)x) dx \\ &= \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \hat{f}_\infty(m).\end{aligned}$$

Conjugating by $\sigma_{\mathfrak{a}}$ we can also prove similarly that

$$\langle f, P_{\mathfrak{a}m}\rangle = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}} \hat{f}_{\mathfrak{a}}(m).$$

(2) Using $\mathcal{F}$ we can write

$$P_{\mathfrak{a}m} = \sum_{f\in\mathcal{F}} \langle P_{\mathfrak{a}m}, f\rangle f.$$

We take $\langle, P_{\mathfrak{b}n}\rangle$ both sides and use the generalised result in (1) to obtain

$$\langle P_{\mathfrak{a}m}, P_{\mathfrak{b}n}\rangle = \sum_{f\in\mathcal{F}} \frac{\Gamma(k-1)^2}{(4\pi\sqrt{mn})^{2k-2}} \hat{f}_{\mathfrak{a}}(m)\overline{\hat{f}_{\mathfrak{b}}(n)}.$$

On the other hand again using 5.1(4) and part (1) above we get that

$$\langle P_{\mathfrak{a}m}, P_{\mathfrak{b}n}\rangle = \frac{\Gamma(k-1)}{(4\pi n)^{k-1}} (n/m)^{\frac{k-1}{2}} \left\{\delta_{\mathfrak{a}\mathfrak{b}}\delta_{mn} + \frac{1}{(2\pi i)^k} \sum_{c>0} \frac{S_{\mathfrak{a}\mathfrak{b}}(m,n;c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right\}.$$

Thus we conclude.

(3) From (2) we see that for $\mathfrak{a} = \mathfrak{b}$ and $m = n$

$$\|P_{\mathfrak{a}m}\|^2 \frac{(4\pi m)^{k-1}}{\Gamma(k-1)} = 1 + \frac{1}{(2\pi i)^k} \sum_{c>0} \frac{S_{\mathfrak{a}\mathfrak{a}}(m,m;c)}{c} J_{k-1}\left(\frac{4\pi m}{c}\right).$$

To show the claim it is thus enough to show that the sum in the RHS of the above has absolute value strictly smaller than 1. Recalling definition of $c(\mathfrak{a},\mathfrak{a})$ and using 4.9(2) we trivially estimate the Kloosterman sum that

$$\sum_{c\le X} \frac{|S_{\mathfrak{a}\mathfrak{a}}(m,m;c)|}{c} \le \frac{X}{C(\mathfrak{a},\mathfrak{a})}.$$

Also for an absolute constant m_0 we have that

$$\left|J_{k-1}\left(\frac{4\pi m}{c}\right)\right| \leq j_0\left(\frac{4\pi m}{c}\right)^{k-1}.$$

Thus using summation by parts (recall from 1.3) we obtain that

$$\begin{aligned}&\frac{1}{(2\pi)^k}\sum_{c\geq c(\mathfrak{a},\mathfrak{a})}\left|\frac{S_{\mathfrak{a}\mathfrak{a}}(m,m;c)}{c}J_{k-1}\left(\frac{4\pi m}{c}\right)\right|\\ &\leq j_0\frac{(4\pi m)^{k-1}}{(2\pi)^k}(k-1)\int_{c(\mathfrak{a},\mathfrak{a})}^{\infty}\frac{x/c(\mathfrak{a},\mathfrak{a})}{x^k}dx\\ &= j_0\frac{(k-1)}{2\pi(k-2)}\left(\frac{2m}{c(\mathfrak{a},\mathfrak{a})}\right)^{k-1}.\end{aligned}$$

Clearly there is an absolute constant m_0 (say $m_0 = j_0^{1/k-1}/2$) such that if $m \leq m_0 c(\mathfrak{a},\mathfrak{a})$ then the above quantity is smaller than 1. Hence the conclusion follows.

5.5. **Bounds of Fourier coefficients.** Let f be a weight $k > 2$ cusp form for $\Gamma_0(q)$ with Fourier expansion at infinity

$$f(z) = \sum_{n=1}^{\infty} a(n)e(nz).$$

Fix k and q.

(1) Prove that

$$\sum_{n\leq N}|a(n)|^2 \ll_f N^k.$$

(2) Prove that without the absolute value the above sum will have lot of cancellation, in fact,

$$\sum_{n\leq N} a(n)e(\alpha n) \ll_f N^{k/2}\log(2N),$$

for any real α. Thus,

$$\sum_{n\equiv a \mod q; n\leq N} a(n) \ll_f N^{k/2}\log(2N).$$

(3) Let for some $1/2 \leq \sigma < 1$ the sum

$$\sum_{c>0} c^{-2\sigma}|S(n,n;c)| \ll_\epsilon n^\epsilon.$$

Prove that, assuming the above,

$$a(n) \ll_{f,\epsilon} n^{k/2-1+\sigma+\epsilon}.$$

5.6. **Solution.** Here we always write $\mathbb{H} \ni z = x + iy$ with $x \in \mathbb{R}$ and $y > 0$.

(1) Recall the Sobolev estimate from the lecture that

$$\|f\|_\infty = \sup_{z \in \mathbb{H}} y^{k/2} |f(z)| \ll_f 1.$$

Thus $f(z) \ll_f y^{-k/2}$. Now from the Fourier expansion of f and using the Parseval's formula we obtain that

$$\sum_{n=1}^{\infty} |a(n)|^2 e^{-4\pi n y} = \int_0^1 |f(z)|^2 dx \ll_f y^{-k},$$

which is true for any $z \in \mathbb{H}$. Choosing $y = N^{-1}$ and dropping all the terms for $n > N$ from the sum we conclude.

(2) From the Fourier expansion of f we can write that

$$a(n) = \int_0^1 f(z) e(-nz) dx.$$

Now for any real α, we multiply the above equation by $e(\alpha n)$, do a change of variable, and sum over $n \leq N$. We obtain that

$$\sum_{n \leq N} a(n) e(\alpha n) = \int_0^1 f(z + \alpha) S_N(z) dx,$$

where

$$S_N(z) := \sum_{0 < n \leq N} e(-nz) = \frac{e(-Nz) - 1}{1 - e(z)} \ll e^{2\pi N y} |1 - e(z)|^{-1}.$$

Again employing the Sobolev estimate that $f(z + \alpha) \ll_f y^{-k/2}$ and checking that

$$\int_0^1 |1 - e(z)|^{-1} dx \ll \int_0^1 \frac{dx}{z} \ll \log(1 + y^{-1}),$$

we obtain that,

$$\sum_{n \leq N} a(n) e(\alpha n) \ll_f y^{-k/2} e^{2\pi N y} \log(1 + y^{-1}),$$

for any $y > 0$. Thus choosing $y = N^{-1}$ we conclude.

For the second part we first note that

$$\frac{1}{q} \sum_{0 \leq a < q} e(an/q) = \delta_{q|n}(n).$$

Hence,

$$\sum_{n\equiv a \mod q; n\leq N} a(n) = \sum_{n\leq N} a(n)\frac{1}{q}\sum_{b \mod q} e((n-a)b/q)$$
$$= \frac{1}{q}\sum_{b \mod q} e(-ab/q)\sum_{n\leq N} a(n)e(nb/q).$$

We conclude employing the bound from the first part in the second sum.

(3) We choose an orthonormal basis $\mathcal{F}$ of $S_k(\Gamma)$ which contains f. The using $\mathfrak{a} = \infty = \mathfrak{b}$ and $m = n$ in the Petersson trace formula in 5.3(2) we get that

$$\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\sum_{g\in\mathcal{F}}|\hat{g}(n)|^2 = 1 + \frac{1}{(2\pi i)^k}\sum_{c>0}\frac{S(n,n;c)}{c}J_{k-1}\left(\frac{4\pi n}{c}\right).$$

We will drop all the terms but the term for $g = f$ in the LHS; thus it will enough to show that the RHS is $O_{f,\epsilon}(n^{2\sigma-1+\epsilon})$ assuming the bound of the Kloosterman sum. Now using that

$$J_{k-1}(x) \ll \min(x^{k-1}, x^{-1/2}) \leq x^{2\sigma-1},$$

and doing summation by parts (see 1.3) we obtain that

$$\sum_{c>0}\frac{|S(n,n;c)|}{c}\left|J_{k-1}\left(\frac{4\pi n}{c}\right)\right| \ll \sum_{c>0}\frac{|S(n,n;c)|}{c}(n/c)^{2\sigma-1} \ll n^{2\sigma-1+\epsilon}.$$

Hence the conclusion follows.

6. Due on 12th December

6.1. **L-function.** There is a vast general theory about attaching a Dirichlet series, which is called (automorphic/Hecke) L-function in some special context, to an object like a modular form. In this exercise we will see some introductory thing on some modular L-function.

Let f be a weight k modular form of full level $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz).$$

We attach a *meromorphic L-function* to f, defined for $\Re(s)$ sufficiently large by the Dirichlet series

$$L(s, f) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

(1) Prove that the defining Dirichlet series of $L(s, f)$ converges absolutely for $\Re(s) > \frac{k+1}{2}$.
(2) Recall that in the very first lecture we obtained the Riemann zeta function multiplied with some Gamma functions by a Mellin transform of the Theta series. Show that,

$$\int_0^{\infty} (f(iy) - a(0))y^s \frac{dy}{y} = \Gamma_{\mathbb{C}}(s)L(s, f),$$

for $\Re(s)$ sufficiently large and $\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s)$.
(3) Let us call the *completed L-function* to be

$$\Lambda(s, f) := \Gamma_{\mathbb{C}}(s)L(s, f).$$

Use the exponential decay property of $f(iy) - a(0)$ at ∞ and 0 to show that $\Lambda(s, f)$ can be continued meromorphically to the full complex plane with simple poles at $s = 0, k$ with residues $-a(0)$ and $i^k a(0)$ respectively.
(4) Using the Fourier expansion $f\mid_w$ for $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ prove that $\Lambda(s, f)$ satisfies the functional equation:

$$\Lambda(s, f) = i^k \Lambda(k - s, f).$$

(5) Let E_k be the k'th Eisenstein series. Calculate $L(s, E_k)$ in terms of the Riemann zeta functions. Manually (without using the functional equation) check that

$$\Lambda(s, E_k) = i^k \Lambda(k - s, E_k).$$

(6) Let f be now a normalized (i.e. first Fourier coefficient is 1) cuspidal Hecke eigenform (i.e. eigenfunction of all Hecke operators) with p-Hecke eigenvalue $\lambda(p)$. Prove that $L(s, f)$ has an Euler product of the form

$$L(s, f) = \prod_p (1 - \lambda(p)p^{-s} + p^{k-1-2s})^{-1}.$$

Hint: For

$$g : \text{Primes} \times \mathbb{Z}_{\geq 0} \to \mathbb{C}$$

prove that

$$\prod_p \sum_{r=0}^{n} g(p,r) = \sum_{n=1}^{\infty} \prod_{p^r || n} g(p,r),$$

provided g is small enough to justify the rearrangements.

6.2. **Solution.**

(1) Using the result from 5.5(1) and Cauchy-Schwartz we get that

$$A(x) := \sum_{n\le x} |a(n)| \ll \sqrt{x \sum_{n\le x} |a(n)|^2} \ll x^{\frac{k+1}{2}}.$$

Now using the summation by parts (1.3) we see that

$$\sum_{n=1}^{x} \frac{a(n)}{n^s} = \frac{A(x)}{x^s} - s\int_1^{\infty} \frac{A(t)}{t^{s+1}} dt \ll x^{\frac{k+1}{2}-\Re(s)} + |s| \int_1^x t^{\frac{k-1}{2}-\Re(s)} dt.$$

Clearly if $\Re(s) > \frac{k+1}{2}$ the above sum is absolutely convergent as $x \to \infty$, proving the claim.

(2) As we may assume that $\Re(s)$ is sufficiently large we can justify interchange of integration and summation below. Thus

$$\begin{aligned}\int_0^{\infty} (f(iy) - a(0)) y^s \frac{dy}{y} &= \int_0^{\infty} \sum_{n=1}^{\infty} a(n) e^{-2\pi n y} y^s \frac{dy}{y} \\ &= \sum_{n=1}^{\infty} a(n) \int_0^{\infty} e^{-2\pi n y} y^s \frac{dy}{y} = (2\pi)^{-s}\Gamma(s) \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.\end{aligned}$$

From 5.6(1) it is clear that $a(n) \ll n^k$ for $n \ge 1$ (check that

$$\sum_{n=1}^{\infty} |a(n)|^2 e^{-4\pi n y} \ll y^{-k},$$

then choose $y = n-1$ after dropping all but the n'th term in the lHS). Thus for $y \ge 1$

$$|f(iy) - a(0)| \le \sum_{n=1}^{\infty} |a(n)| e^{-2\pi n y} \ll e^{-2\pi y} \sum_{n=1}^{\infty} n^k e^{-2\pi(n-1)} \ll e^{-2\pi y}.$$

Similarly for $y < 1$,

$$\begin{aligned}|f(iy) - (iy)^{-k} a(0)| &= |f(i/y)(iy)^{-k} - (iy)^{-k} a(0)| \\ &= y^{-k} \sum_{n=1}^{\infty} |a(n)| e^{-2\pi n/y} \\ &\ll e^{-\pi/y}.\end{aligned}$$

For $\Re(s) > k+1$ large enough we have the following:

$$\begin{aligned}\Lambda(s,f) &= \int_0^\infty (f(iy)-a(0))y^s\frac{dy}{y}\\ &= \int_0^1 (f(iy)-(iy)^{-k}a(0))y^s\frac{dy}{y} + \int_1^\infty (f(iy)-a(0))y^s\frac{dy}{y}\\ &\qquad + a(0)\int_0^1((iy)^{-k}-1)y^s\frac{dy}{y}\\ &= h(s) + \frac{i^k a(0)}{s-k} - \frac{a(0)}{s}.\end{aligned}$$

where

$$h(s) := \int_0^1 (f(iy)-(iy)^{-k}a(0))y^s\frac{dy}{y} + \int_1^\infty (f(iy)-a(0))y^s\frac{dy}{y}$$

is entire by the previous two estimates of $f(iy)$. Thus $\Lambda(s,f)$ has simple poles at $s=0,k$ with residues $-a(0)$ and $i^k a(0)$ respectively. Now we see that

$$\begin{aligned}h(k-s) &= \int_0^1 (f(iy)-(iy)^{-k}a(0))y^{k-s}\frac{dy}{y} + \int_1^\infty (f(iy)-a(0))y^{k-s}\frac{dy}{y}\\ &= \int_1^\infty (f(i/y)y^{-k}-i^k a(0))y^s\frac{dy}{y} + \int_0^1 (f(i/y)y^{-k}-a(0)y^{-k})y^s\frac{dy}{y}\\ &= \int_0^1 (f(iy)i^k - a(0)y^{-k})y^s\frac{dy}{y} + i^k\int_1^\infty (f(iy)-a(0))y^s\frac{dy}{y}\\ &= i^k h(s).\end{aligned}$$

This proves the functional equation: $\Lambda(k-s,f) = i^k\Lambda(s,f)$.

(3) Recalling the Fourier series of E_k from 3.3 we get that

$$\begin{aligned}L(s,E_k) &= \sum_{n=1}^\infty \frac{\sigma_{k-1}(n)}{n^s} = \sum_{n=1}^\infty n^{-s}\sum_{d|n} d^{k-1}\\ &= \sum_{n,m=1}^\infty n^{-s}m^{-s+k-1} = \zeta(s)\zeta(s-k+1).\end{aligned}$$

Now using the functional equation of the Riemann zeta function we get that

$$\zeta(s-k+1) = \pi^{s-k+\frac{1}{2}}\frac{\Gamma(\frac{k-s}{2})}{\Gamma(\frac{1-k+s}{2})}\zeta(k-s).$$

Thus using the duplication formula

$$\Gamma(s) = 2^{s-1}\pi^{-1/2}\Gamma(s/2)\Gamma(s/2+1/2),$$

and using notation that

$$(a)_k := (a-1)\dots(a-k) = \frac{\Gamma(a)}{\Gamma(a-k)},$$

we get that

$$\begin{aligned}\Lambda(s, E_k) &= (2\pi)^{-s}\Gamma(s)\zeta(s)\pi^{s-k+\frac{1}{2}}\frac{\Gamma(\frac{k-s}{2})}{\Gamma(\frac{1-k+s}{2})}\zeta(k-s)\\ &= \frac{1}{2\pi^{k/2}}\left(\frac{s+1}{2}\right)_{k/2}\Lambda(s)\Lambda(k-s),\end{aligned}$$

where $\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$. Now we check that

$$\left(\frac{k-s+1}{2}\right)_{k/2} = \prod_{r=1}^{k/2}\frac{k-2r-s+1}{2} = \prod_{r=1}^{k/2}\frac{2r-s-1}{2} = (-1)^{k/2}\left(\frac{s+1}{2}\right)_{k/2}.$$

This concludes the proof.

(4) From the Hecke multiplicativity we remember that for p prime,

$$\lambda(p^r)\lambda(p) = \lambda(p^{r+1}) + p^{k-1}\lambda(p^{r-1}); \quad \forall r \geq 1, \quad \lambda(1) = 1.$$

We first note that,

$$\begin{aligned}&(1-\lambda(p)p^{-s}+p^{k-1-2s})\sum_{r=0}^{\infty}\frac{\lambda(p^r)}{p^{rs}}\\ &= \sum_{r=0}^{\infty}(\lambda(p^r) - p^{-s}\lambda(p^{r+1}) - p^{k-1-s}\lambda(p^{r-1}) + p^{k-1-2s}\lambda(p^r))p^{-rs}\\ &= \sum_{r=0}^{\infty}\frac{\lambda(p^r)}{p^{rs}} - \sum_{r=1}^{\infty}\frac{\lambda(p^r)}{p^{rs}} - p^{k-1-2s}\sum_{r=0}^{\infty}\frac{\lambda(p^r)}{p^{rs}} + p^{k-1-2s}\sum_{r=0}^{\infty}\frac{\lambda(p^r)}{p^{rs}}\\ &= \lambda(1) = 1.\end{aligned}$$

Thus to prove the claim it is enough to show that

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = \prod_p\sum_{r=0}^{\infty}\frac{\lambda(p^r)}{p^{rs}}.$$

Assuming that $\Re(s)$ is large enough so that the sum in consideration converges absolutely we can freely rearrange the summands. Now we recall that

$$\lambda(m)\lambda(n) = \lambda(mn); \quad (m,n) = 1.$$

From fundamental theorem of arithmetic one can write that

$$n = \prod_{p^r || n} p^r.$$

Thus,

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s}=\sum_{n=1}^{\infty}\frac{\lambda(\prod_{p^r||n}p^r)}{\prod_{p^r||n}p^{rs}}=\sum_{n=1}^{\infty}\prod_{p^r||n}\frac{\lambda(p^r)}{p^{rs}}=\prod_{p}\sum_{r=0}^{\infty}\frac{\lambda(p^r)}{p^{rs}}.$$

This completes the proof.

6.3. **Multiplicity one principle.** Let χ be a primitive Dirichlet character mod q. Check from the definition that $S_k(\Gamma_0(q),\chi)$ is the space of newforms.

(1) For any positive integer d let us define the operator

$$A_d := \frac{1}{d}\sum_{b \mod d}\begin{pmatrix}1 & b/d\\ & 1\end{pmatrix}.$$

Also define

$$S_q := \sum_{d|q}\mu(d)A_d.$$

Show that if $f\in S_k(\Gamma_0(q),\chi)$ having Fourier expansion

$$f(z)=\sum_{n=1}^{\infty}a(n)e(nz),$$

then

$$S_qf(z)=\sum_{(n,q)=1}a(n)e(nz).$$

(2) Prove that if q is square-free then S_q is injective.
(3) Using the above show that if q is square-free and if f is eigenfunction of Hecke operators T_n with $(n,q)=1$ then it is eigenfunction of T_n for all $n\in\mathbb{N}$.

6.4. **Solution.**

(1) We check that

$$\begin{aligned}A_df(z)&=\frac{1}{d}\sum_{b \mod d}f(z+b/d)\\&=\sum_{n=1}^{\infty}a(n)e(nz)\frac{1}{d}\sum_{b \mod d}e(nb/d)=\sum_{d|n}a(n)e(nz).\end{aligned}$$

Thus, as

$$\sum_{d|n}\mu(d)=\delta_{n=1},$$

we get

$$S_q f(z) = \sum_{d|q} \mu(d) \sum_{d|n} a(n)e(nz)$$
$$= \sum_{n=1}^{\infty} a(n)e(nz) \sum_{\substack{d|q \\ d|n}} \mu(d) = \sum_{(n,q)=1} a(n)e(nz).$$

This completes the proof.

(2) To prove S_q has zero kernel it is enough to show that if $a(n) = 0$ for all $(n, q) = 1$ then $f = 0$ i.e. $a(n) = 0$ for all n.

Suppose that $\tau = \begin{pmatrix} a & * \\ c & d \end{pmatrix} \in \Gamma_0(q)$. Let us recall the modularity relation of f at the point $\frac{z-d}{c}$. We get that,

$$z^{-k} f\left(\frac{a}{c} - \frac{1}{cz}\right) = \chi(d) f\left(\frac{z-d}{c}\right).$$

Expanding both sides into corresponding Fourier series.

$$z^{-k} \sum_{m=1}^{\infty} a(m) e\left(\frac{am}{c} - \frac{m}{cz}\right) = \chi(d) \sum_{n=1}^{\infty} a(n) \left(\frac{n(z-d)}{c}\right).$$

Now we take $c = q$ and $ad \equiv 1(\mod c)$. Hence , summing over $a \in \mathbb{Z}/q\mathbb{Z}^{\times}$, we get that,

$$z^{-k} \sum_{m=1}^{\infty} a(m) S(m, 0; q) e(-m/qz) = \sum_{n=1}^{\infty} a(n) \bar{\chi}(-n) \tau(\chi) e(nz/q),$$

where $S(m, 0; q)$ is the Ramanujan sum and $\tau(\chi)$ is the Gauss sum. We see that every term in the RHS is zero by the hypothesis; hence so is every term in the LHS. Therefore for all $m \geq 1$

$$a(m) S(m, 0; q) = 0.$$

Now as q is square-free then Ramanujan sum

$$S(m, 0; q) = \mu(q/(m, q)) \phi((m, q)) \neq 0,$$

This completes the proof.

(3) Part (2) implies that the n'th Hecke eigenvalues for $(n, q) = 1$ determines f completely. Now we know that for any m the Hecke operators T_m and T_n commute. Thus for $(n, q) = 1$ and f Hecke eigenfunction of T_n with eigenvalue $\lambda(n)$ we see that

$$T_n(T_m f) = T_m(T_n f) = T_m(\lambda(n) f) = \lambda(n)(T_m f).$$

That is, $T_m f$ has same Hecke eigenvalues for T_n for $(n, q) = 1$. Thus $T_m f$ must be a constant multiple of f, which prove the claim.

7. Due on 19th December

7.1. Jacobi's Product Formula. Recall the standard (Jacobi) theta function.

$$\theta(z) := \sum_{n\in\mathbb{Z}} e(n^2 z/2).$$

We will prove Jacobi's product formula:

$$\theta(z) = \prod_{n=1}^{\infty}(1-e(2nz))(1-e((2n+1)z))^2.$$

(1) For $z\in\mathbb{H}$ and $w\in\mathbb{C}$ define

$$\Theta(w \mid z) := \sum_{n\in\mathbb{Z}} e(n^2 z/2 + nw),$$

and

$$\Pi(w \mid z) := \prod_{n=1}^{\infty}(1-e(nz))(1+e((n-1/2)z+w))(1+e((n-1/2)z-w)).$$

Show that Π and Θ are entire functions of w. Also show that they have zeros when $w = n+1/2+z(m+1/2)$ for $m,n\in\mathbb{Z}$, and they are the only zeros of Π and they are simple.

(2) Define

$$F(w \mid z) = \frac{\Theta(w \mid z)}{\Pi(w \mid z)}.$$

Prove that F, as a function of w, is doubly periodic. Conclude that F, as a function of w, is constant, say $c(z)$.

(3) Calculating F at $w=1/2$ and $w=1/4$ conclude that

$$c(z) = c(4z).$$

(4) Conclude that $c(z)=1$ and thus obtain Jacobi's product formula.

7.2. Solution.

(1) To see that Θ is entire (in z and w) it is enough to show uniform convergence in compact subsets. Let $|w|\leq M$ and $\Im(z)\geq y$ then

$$\sum_{n\in\mathbb{Z}} e(n^2 z/2+nw) \leq \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y + 2\pi n M} \ll_{M,y} 1.$$

To see that Π is entire we again suppose that $|w|\leq M$ and $\Im(z)\geq y$. Then

$$(1-e(nz))(1+e((n-1/2)z+w))(1+e((n-1/2)z-w)) = 1+O_{M,y}(e^{-2\pi n y}).$$

Again from standard result about convergence of product and noting that

$$\sum_{n=1}^{\infty} e^{-2\pi n y} \ll 1,$$

we conclude.

To see the zeros of Θ we first note that

$$\begin{aligned}\Theta(w+z \mid z) &= \sum_{n\in\mathbb{Z}} e((n^2+2n)z/2+nw) \\ &= e(-z/2-w)\sum_{n\in\mathbb{Z}} e((n+1)^2 z/2+(n+1)w) \\ &= e(-z/2-w)\Theta(w \mid z).\end{aligned}$$

Because of this relation and that Θ is 1-periodic in w it is enough to show that Θ has zero for $w = 1/2 + z/2$. To see this we check that

$$\begin{aligned}\Theta(1/2+z/2 \mid z) &= \sum_{n\in\mathbb{Z}} e((n^2+n)z/2)e^{\pi i n} \\ &= \sum_{n=0}^{\infty}(-1)^n e((n^2+n)z/2) + \sum_{n=0}^{\infty}(-1)^{n+1} e(((n+1)^2-(n+1))z/2) \\ &= \sum_{n=0}^{\infty}(-1)^n (e(n^2+n)z/2 - e((n^2+n)z/2)) = 0.\end{aligned}$$

The only zeros of Π should come from its factors. As $z \in \mathbb{H}$ we have $|e(nz)| < 1$. So only zeros are possible from other two type of factors. In fact, Π has a zero only if

$$w \in z(n+1/2) + \mathbb{Z}.$$

Again as $e(x) - 1$ has simple zero at $x \in \mathbb{Z}$ the conclusion follows.

(2) We note that

$$\begin{aligned}\Pi(w+z \mid z) &= \prod_{n=1}^{\infty}(1-e(nz))(1+e((n+1/2)z+w))(1+e((n-3/2)z-w)) \\ &= \frac{1-e(-z/2-w)}{1-e(z/2+w)}\Pi(w+z \mid z) \\ &= e(-z/2-w)\Pi(z+w \mid z).\end{aligned}$$

Thus it is clear that F is doubly periodic with period 1 and z as a function of w. As Θ and Π are entire and zeros of Π get cancelled by zeros of Θ we can also conclude F is entire as function of w. As any doubly periodic entire function on complex plane is constant by Liouville's theorem we conclude.

(3) We put $w = 1/2$ and check that

$$\begin{aligned}\sum_{n\in\mathbb{Z}}(-1)^n e(n^2z/2) &= c(z)\prod_{n=1}^{\infty}(1-e(nz))(1-e((n-1/2)z))^2\\ &= c(z)\prod_{n=1}^{\infty}(1-e(nz/2))(1-e((n-1/2)z)).\end{aligned}$$

We put $w = 1/4$ and obtain

$$\begin{aligned}\sum_{n\in\mathbb{Z}} i^n e(n^2z/2) &= \sum_{n\in\mathbb{Z}}(-1)^n e(2n^2z)\\ &= c(z)\prod_{n=1}^{\infty}(1-e(nz))(1+ie((n-1/2)z))(1-ie((n-1/2)z))\\ &= c(z)\prod_{n=1}^{\infty}(1-e(nz))(1+e((2n-1)z))\\ &= c(z) = \prod_{n=1}^{\infty}(1-e(2nz))(1-e((2n-1)2z)).\end{aligned}$$

From previous two equalities we conclude that $c(4z) = c(z)$.

(4) From previous part we conclude that for any $k \geq 0$ we have $c(4^kz) = c(z)$. Thus

$$c(z) = \lim_{k\to\infty}\frac{\Theta(1 \mid 4^kz)}{\Pi(1 \mid 4^kz)} = 1.$$

The last equality follow form $\lim_{k\to\infty} e(4^kz) = 0$. Putting $w = 0$ we obtain Jacobi's product formula.

7.3. **Jacobi's 8-squares Formula.** Recall that

$$r_8(n) := |\{(x_1,\dots,x_8)\in\mathbb{Z}^8 \mid x_1^2+\dots+x_8^2 = n\}|.$$

In this exercise we will prove an exact formula of $r_8(n)$ due to Jacobi. To do that we need some preparation. Define a level 2 congruence subgroup

$$\Gamma := \langle T^2, S\rangle = \langle\begin{pmatrix}1 & 2\\ & 1\end{pmatrix},\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\rangle \subset \mathrm{SL}_2(\mathbb{Z}).$$

It can be checked that Γ has two cusps ∞ and $TS\infty$. Consider the space $M_k(\Gamma)$ of weight k modular forms on $\Gamma\backslash\mathbb{H}$ with usual definition.

(1) Prove that there is no weight zero non-constant modular form, i.e. $M_0(\Gamma) = \mathbb{C}$.

(2) Consider Jacobi's θ-function as defined in the previous exercise. Check that

$$\lim_{\Im(z)\to\infty}\sqrt{i/z}\,\theta(1-1/z)e(-z/8) = 2.$$

(3) Prove that θ does not vanish in $\mathbb{H}\cup\{\infty\}$.

(4) Let $4 \mid k$. Let $f \in M_{\frac{k}{2}}(\Gamma)$ such that

$$\lim_{\Im(z)\to\infty} (i/z)^{k/2} f(1-z^{-1})e(-kz/8)$$

exists. Prove that $f = c\theta^k$ for some $c \in \mathbb{C}$.

(5) Recall the Eisenstein series

$$G_k(z) := \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} (mz+n)^{-k}$$

and consider $a, b \in \mathbb{C}$. Prove that $f(z) := aG_k(z) + bG_k(\frac{z+1}{2}) \in M_k(\Gamma)$. Moreover, show that if $k = 4$ and $a = -16b$, then $f = c\theta^8$ for some $c \in \mathbb{C}$.

(6) Show that,

$$\theta^8(z) = \frac{3}{\pi^4}\left(16G_4(z) - G_4\left(\frac{z+1}{2}\right)\right),$$

and conclude that

$$r_8(n) = 16\sum_{d\mid n}(-1)^{n-d}d^3.$$

Optional:* Using similar technique find exact formulas for r_2 and r_4.

7.4. **Solution.**

(1) Let $f \in M_0(\Gamma)$. Note that the fundamental domain $\Gamma\backslash\mathbb{H}$ has two cusps ∞ and $TS\infty = 1$. We define

$$g(z) = (f(z) - f(\infty))(f(z) - f(1)).$$

As f is invariant by Γ, so is g. But also g is bounded in $\overline{\Gamma\backslash\mathbb{H}}$ thus by invariance is bounded in $\mathbb{H}$. Thus by maximum modulus principal g is constant, thus must be 0 as $g(1) = 0$. So $f(z)$ has only two possibilities. But f is continuous, and as $\mathbb{H}$ is connected, thus can only be constant.

(2) Recall the transformation law of the θ function: for $w \in \mathbb{C}$ and $z \in \mathbb{H}$ one has

$$\sqrt{\frac{z}{i}}\sum_{n\in\mathbb{Z}} e((n+w)^2 z/2) = \sum_{n\in\mathbb{Z}} e(-n^2/2z + nw).$$

Putting $w = 1/2$ we get that

$$\begin{aligned}\sqrt{\frac{z}{i}}\sum_{n\in\mathbb{Z}} e((n+1/2)^2 z/2) &= \sum_{n\in\mathbb{Z}} e(-n^2/2z + n/2)\\ &= \sum_{n\in\mathbb{Z}} e(-n^2/2z + n^2/2)e(-n^2/2 + n/2)\\ &= \sum_{n\in\mathbb{Z}} e((1-1/z)n^2/2) = \theta(1-1/z).\end{aligned}$$

Thus,

$$\sqrt{i/z}\theta(1-1/z)e(-z/8) = e(-z/8)\left(2e(z/8) + \sum_{n\neq 0,-1} e((n+1/2)^2 z/2)\right).$$

Conclusion follows from the fact that $\lim_{\Im(z)\to\infty} e(z) = 0$.

(3) From the definition of the theta function we immediately see that

$$\theta(\infty) = 1.$$

We will prove that θ does not vanish in $\mathbb{H}$. But due to the modular transformation law it is enough to show that θ does not vanish in $\Gamma\backslash\mathbb{H}$. From the definition of Γ one can check that

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma \cup \Gamma T \cup \Gamma TS.$$

Thus the fundamental domains of $\mathrm{SL}_2(\mathbb{Z})$ and Γ are related by

$$\mathcal{F}_\Gamma = \mathcal{F}_{\mathrm{SL}_2(\mathbb{Z})} \cup T\mathcal{F}_{\mathrm{SL}_2(\mathbb{Z})} \cup TS\mathcal{F}_{\mathrm{SL}_2(\mathbb{Z})}.$$

Hence it suffices to show that

$$\theta(z), \theta(z+1), \theta(1-1/z) \neq 0, \quad z \in \mathcal{F}_{\mathrm{SL}_2(\mathbb{Z})}.$$

We may assume that $\Im(z) \geq \sqrt{3}/2$ from the description of standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$. Note that

$$\begin{aligned}|\theta(z) - 1| &\leq \sum_{n\neq 0} e^{-\pi n^2 y}\\ &\leq 2\sum_{n=1}^{\infty} e^{-n\pi\sqrt{3}/2} < 0.2.\end{aligned}$$

Thus $|\theta(z)| > 0.8$. Similarly, $|\theta(1+z)| > 0.8$. To show that $\theta(1-1/z) \neq 0$ from the proof of (2) it suffices to prove that

$$\sum_{n\in\mathbb{Z}} e((n^2+n)z/2) \neq 0, \quad \Im(z) \geq \sqrt{3}/2.$$

We note that

$$\begin{aligned}|\sum_{n\in\mathbb{Z}} e((n^2+n)z/2) - 2| &\leq \sum_{n=1}^{\infty} e^{-n(n+1)\pi\sqrt{3}/2}\\ &\leq \sum_{n=1}^{\infty} e^{-n\pi\sqrt{3}/2} < 0.2.\end{aligned}$$

Thus $|\sum_{n\in\mathbb{Z}} e((n^2+n)z/2)| > 1.8$, and we conclude.

(4) It is enough to show that $f/\theta^k \in M_0(\Gamma)$ as the claim follows by (1). By definition f and θ^k satisfy modular transform of weight $k/2$. From part (3) it is also clear that f/θ^k is holomorphic in $\mathbb{H} \cup \{\infty\}$. Thus it is enough to show that f/θ^k is holomorphic at the cusp $TS\infty$. From the assumption it suffices to show that

$$\lim_{\Im(z)\to\infty} (i/z)^{k/2}\theta^k(1-1/z)e(-kz/8)$$

exists and nonzero. But from (2) this limit is 2^k, thus we conclude.

(5) Because $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ congruence we have $M_k(\Gamma) \supseteq M_k(\mathrm{SL}_2(\mathbb{Z})$. Thus $G_k(z) \in_k (\Gamma)$. We will show that $G'(z) := G_k(\frac{z+1}{2}) \in M_k(\gamma)$. As

$$G'(\infty) = \lim_{\Im(z)\to\infty} G'(z) = \lim_{\Im(z)\to\infty} G_k(z) = 2\zeta(k),$$

Also we see that

$$G'(1) = \lim_{\Im(z)\to\infty} z^{-k}G'(1-1/z) = 2^k \lim_{\Im(z)\to\infty} G_k(2z) = 2^{k+1}\zeta(k).$$

we conclude that G' is holomorphic in $\mathbb{H} \cup \{\infty\}$. Thus to show modularity we need to show that G' satisfies the modular transformations under Γ. Clearly,

$$G'(T^2z) = G'(z+2) = G_k(1+(z+1)/2) = G'(z).$$

Now

$$G'(Sz) = G_k(1/2 - 1/2z) = \sum_{(m,n)\neq(0,0)} \left(\frac{2z}{(m-1)z+2n}\right)^k$$

$$= z^k \sum_{(m,n)\neq(0,0)} \left(\frac{2}{(m+1)z+2n}\right)^k = z^k G_k((z+1)/2) = z^k G'(z).$$

Thus a linear combination of G_k and G' is an element in $M_k(\Gamma)$.

Now we need to show that, WOLG taking $b=1$ that

$$G(z) := 16G_4(z) - G_k((z+1)/2) \in \mathbb{C}\theta^8.$$

but from (4) it is enough to show that $\lim_{\Im(z)\to\infty} z^{-4}G(1-1/z)e(-z/2)$ exists. By modularity we see that

$$\begin{aligned}
&\lim_{\Im(z)\to\infty} z^{-4}G(1-1/z)e(-z) \\
&= \lim_{\Im(z)\to\infty} (16G_4(z) - 2^4G_4(2z))e(-z) \\
&= 16.2.\zeta(4) \lim_{\Im(z)\to\infty} 240. \sum_{n=1}^{\infty} \sigma_3(n)(e(n-1)z - e((2n-1)z)) \\
&= 16.2.\zeta(4).240 = \frac{2^8\pi^4}{3}.
\end{aligned}$$

Thus we also can conclude $G = \frac{\pi^4}{3}\theta^8$.

(6) The first part is already shown in (5). From the definition of θ we thus obtain that

$$1+\sum_{n=1}^{\infty} r_8(n)e(nz/2) = 1+16^2\sum_{n=1}^{\infty}\sigma_3(n)e(nz) - 16\sum_{n=1}^{\infty}\sigma_3(n)(-1)^n e(nz/2).$$

Comparing the Fourier coefficients we conlude that

$$r_8(n) = 16\sigma_3(n) = 16\sum_{d|n}(-1)^{n-d}d^3,$$

for odd n. For even n

$$\begin{aligned} r_8(n) &= 16^2\sigma_3(n/2) - 16\sigma_3(n) = 16^2\sum_{2d|n} d^3 - 16\sum_{d|n} d^3 \\ &= 16\left(\sum_{2|d|n} 2d^3 - \sum_{d|n} d^3\right) = 16\left(\sum_{2|d|n} d^3 - \sum_{2\nmid d|n} d^3\right) \\ &= 16\sum_{d|n}(-1)^{n-d}d^3. \end{aligned}$$

This concludes the proof.