

MODULAR FORMS EXERCISES AND SOLUTIONS

1. DUE ON 26TH SEPTEMBER

1.1. **Exercise.** Let \mathcal{P} be the set of primes. Prove that $\sum_{p \in \mathcal{P}} \frac{1}{p} = +\infty$.

1.2. **Solution.** Let $s > 1$. Then from the Euler product of the Zeta function,

$$\begin{aligned} \log \zeta(s) &= \sum_{p \in \mathcal{P}} -\log(1 - p^{-s}) = \sum_{p \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} \\ &\leq \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \sum_{k=2}^{\infty} \frac{1}{p^k} = \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \frac{1}{p(p-1)} \\ &= \sum_{p \in \mathcal{P}} \frac{1}{p^s} + O(1) \end{aligned}$$

As we know that $\lim_{s \rightarrow 1+} \zeta(s) = +\infty$, letting $s \rightarrow 1+$ in the above inequality we conclude that

$$\lim_{s \rightarrow 1+} \sum_{p \in \mathcal{P}} \frac{1}{p^s} = +\infty,$$

hence the result.

1.3. **Summation by Parts.** Let $a : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function, let $0 < y < x$ and let $f : [y, x] \rightarrow \mathbb{C}$ be a function with continuous derivative on $[y, x]$. Then

$$\sum_{y < n \leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where $A(x) = \sum_{n \leq x} a_n$.

1.4. **Exercise.** Prove that for every $\delta > 0$,

$$\pi(x) := |\{p \in \mathcal{P} \mid p \leq x\}|$$

is bigger than $\frac{x}{(\log x)^{1+\delta}}$ for some sufficiently large x .

1.5. **Solution.** Let a_n be the prime indicator function, i.e.

$$a_n := \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not a prime.} \end{cases} .$$

Using summation by parts we note that,

$$\sum_{p \leq x} \frac{1}{p} = \sum_{3/2 < n \leq x} \frac{a_n}{n} = \frac{\pi(x)}{x} + \int_{3/2}^x \frac{\pi(t)}{t^2} dt.$$

If the claim is false i.e. for all sufficiently large x , $\pi(x) \leq x/(\log x)^{1+\delta}$ then from the above,

$$\sum_{p \leq x} \frac{1}{p} \leq \frac{1}{(\log x)^{1+\delta}} + C + \frac{1}{(\log x)^\delta},$$

for some constant C . The RHS of the above tends to C as $x \rightarrow \infty$ contradicting Exercise 1.1, hence the result.

1.6. **Exercise.** Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\}$ is the fractional part of x . Using this show that $\zeta(s)$ has meromorphic continuation to $\Re(s) > 0$ with a simple pole at $s = 1$.

1.7. **Solution.** Let $\Re(s) > 1$. Then using the summation by parts as following.

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n^s} &= \frac{[x]}{x^s} + s \int_1^x \frac{[t]}{t^{s+1}} dt = \frac{1}{x^{s-1}} - \frac{\{x\}}{x^s} + s \int_1^x t^{-s} dt - s \int_1^x \frac{\{t\}}{t^{s+1}} dt \\ &= \frac{s}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}} dt + O(x^{-\Re(s)} + x^{-\Re(s)+1}). \end{aligned}$$

Letting $x \rightarrow \infty$, as $\Re(s) > 1$, we conclude that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$

We now note that the integral right hand side is well defined for $\Re(s) > 0$ and is holomorphic in s . As $\frac{s}{s-1}$ is a meromorphic function with simple pole at $s = 1$ and residue 1, we conclude the meromorphic continuation of $\zeta(s)$ to $\Re(s) > 0$.

1.8. **Exercise.** Prove that the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt$$

has analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple pole at each non-positive integer. Find the residues of the Gamma function at those poles.

Hint: First prove that $\Gamma(s+1) = s\Gamma(s)$.

1.9. **Solution.** By integration by parts we see that

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1} \frac{dt}{t} = \int_0^\infty e^{-t} s t^s \frac{dt}{t} = s\Gamma(s),$$

for $\Re(s) > 0$. Thus $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ extends definition of $\Gamma(s)$ to $\Re(s) > -1$ meromorphically with pole at $s = 0$ as

$$\lim_{s \rightarrow 0^+} \int_0^\infty e^{-t} t^s \frac{dt}{t} = +\infty.$$

The pole is simple, as $\lim_{s \rightarrow 0} s\Gamma(s) = 1$, and with residue 1. Similarly $\Gamma(s)$ can be extended to all $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple poles at $s = -n$, $n \in \mathbb{N}$ with residue,

$$\lim_{s \rightarrow -n} (s+n)\Gamma(s) = \lim_{s \rightarrow -n} \frac{\Gamma(s+n+1)}{(s+n-1)\dots s} = \frac{(-1)^n}{n!}.$$

1.10. **Exercise.** Prove the Poisson summation formula: Let $f \in \mathcal{S}(\mathbb{R})$ be in the Schwartz class. Prove that

$$\sum_{n \in \mathbb{Z}} f(n+u) = \sum_{n \in \mathbb{Z}} \hat{f}(n)e(nu).$$

Note: Putting $u = 0$ we get the usual Poisson summation formula.

1.11. **Solution.** Let

$$F(x) := \sum_{n \in \mathbb{Z}} f(n+x)$$

which is a function on $L^1(\mathbb{R}/\mathbb{Z})$ so has a Fourier expansion of the form

$$F(x) = \sum_{n \in \mathbb{Z}} e(nx)\hat{F}(n).$$

Here

$$\begin{aligned} \hat{F}(n) &= \int_0^1 F(x)e(-nx)dx = \sum_{m \in \mathbb{Z}} \int_0^1 \sum_{m \in \mathbb{Z}} f(m+x)e(-nx)dx \\ &= \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(x)e(-nx)dx = \int_{-\infty}^\infty f(x)e(-nx)dx = \hat{f}(n), \end{aligned}$$

this provides the result.

1.12. **Exercise.** Recall that,

$$G(1, N) := \sum_{n \pmod N} e(n^2/N).$$

Prove that

- (1) For any odd positive integer N , $G(1, N^2) = N$ and $G(1, N^3) = NG(1, N)$.
- (2) For every positive integer N , $G(1, N) = \frac{1+i^{-N}}{1-i} \sqrt{N}$.

1.13. **Solution.** (1) is elementary. We can parametrize the residue class of N^k by

$$\{a_1 N^{k-1} + a_2 N^{k-2} + \cdots + a_k \mid 0 \leq a_i \leq N-1\}.$$

Using this we have,

$$\begin{aligned} G(1, N^2) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(aN+b)^2}{N^2}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \sum_{a=0}^{N-1} e\left(\frac{2ab}{N}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \delta_{b=0} N = N. \end{aligned}$$

Similarly,

$$\begin{aligned} G(1, N^3) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{c=0}^{N-1} e\left(\frac{(aN^2 + bN + c)^2}{N^3}\right) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) \sum_{a=0}^{N-1} e(2ac/N) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) N \delta_{c=0} = NG(1, N). \end{aligned}$$

For the second part we use the Poisson summation formula. First we note the function

$$f(x) := 1_{[0, N]} e(x^2/N)$$

is a function which is continuous on $(0, N)$ and has continuity only from one side at $x = 0, N$. From the Fourier theory we know that the Fourier series of f at $x = 0$ would converge to $\frac{f(0+) + f(0-)}{2} = f(0+)/2$. and similarly, at $x = N$ to $f(N-)/2$. Thus using the (modified) Poisson summation formula and using that $f(0+) = f(N-)$ we get that,

$$\begin{aligned} \sum_{n=0}^N e(N^2/N) &= \frac{f(0+)}{2} + \sum_{n=1}^{N-1} f(n) + \frac{f(N-)}{2} \\ &= \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(nx) dx = \sum_{n \in \mathbb{Z}} \int_0^N e(x^2/N + nx) dx. \end{aligned}$$

Thus,

$$G(1, N) = N \sum_{n \in \mathbb{Z}} \int_0^1 e(Nx^2 + nNx) dx = N \sum_{n \in \mathbb{Z}} e(-Nn^2/4) \int_0^1 e(N(x + n/2)^2)$$

Noting that

$$e(-Nn^2/4) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ i^{-N}, & \text{if } n \text{ is odd.} \end{cases}$$

and dividing the above sum into odd and even parts we get that,

$$\begin{aligned} G(1, N) &= N \sum_{n \in \mathbb{Z}} \int_n^{1+n} e(Nx^2) dx + Ni^{-N} \sum_{n \in \mathbb{Z}} \int_{n-1/2}^{n+1/2} e(nx^2) dx \\ &= \sqrt{N}(1 + i^{-N}) \int_{-\infty}^{\infty} e(y^2) dy. \end{aligned}$$

The last integral can be checked convergent and we call it C . Thus,

$$G(1, N) = \sqrt{N}C(1 + i^{-N}).$$

Checking that, $G(1, 1) = 1$, we conclude the result.

1.14. Dirichlet Character. A *Dirichlet character with modulus q* is a character

$$\chi : \mathbb{Z}/q\mathbb{Z}^\times \rightarrow \mathbb{C}^\times$$

extended to \mathbb{Z} by making it q -periodic and defining $\chi(a) = 0$ for $(a, q) > 1$. Associated to each character χ , in addition to its modulus q , is a natural number q' , its conductor. The *conductor q'* is the smallest divisor of q such that χ can be written as $\chi = \chi' \chi_0$, where χ_0 is the trivial Dirichlet character mod q and χ' is a character of modulus q' . If a character has conductor equal to its modulus then it is called a *primitive Dirichlet character*. Check that, for a primitive Dirichlet character χ mod q one has

$$\frac{1}{q} \sum_{a \pmod q} \chi(ma + b) = \begin{cases} \chi(b), & \text{if } q \mid m \\ 0, & \text{if } q \nmid m. \end{cases}$$

The above is not true for a non-primitive character.

1.15. Exercise. Let χ be a primitive Dirichlet character mod q and $f \in L^1(\mathbb{R})$. Prove that

$$\sum_{m \in \mathbb{Z}} f(m) \chi(m) = \frac{G(\chi)}{q} \sum_{n \in \mathbb{Z}} \hat{f}(n/q) \bar{\chi}(n),$$

where $G(\chi)$ is the Gauss sum attached to χ defined by

$$G(\chi) := \sum_{a \pmod q} \chi(a) e(a/q).$$

Hint: Use the Poisson summation formula.

1.16. **Solution.** First we prove the following. Let $v \in \mathbb{R}$ and $u \in \mathbb{R}^+$. Then using the Poisson summation formula,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} f(um + v) &= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(ux + v) e(-mx) dx \\ &= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(-m(x - v)/u) \frac{dx}{u} \\ &= \frac{1}{u} \sum_{m \in \mathbb{Z}} \hat{f}(m) e(mv/u). \end{aligned}$$

Using the above we get that,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} f(m) \chi(m) &= \sum_{m \in \mathbb{Z}} \sum_{a \pmod q} \chi(a) f(mq + a) \\ &= \sum_{a \pmod q} \chi(a) \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}(m) e(ma/q) \\ &= \frac{G(\chi)}{q} \sum_{m \in \mathbb{Z}} \hat{f}(m) \bar{\chi}(m). \end{aligned}$$

Here in the last line we have used that for a primitive Dirichlet character χ ,

$$\sum_{a \pmod q} \chi(a) e(am/q) = \bar{\chi}(m) G(\chi).$$

This can be seen as follows. Let $(m, q) = 1$. Then,

$$\bar{\chi}(m) G(\chi) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(am^{-1}) e(a/q) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) e(am/q).$$

If $(m, q) > 1$ then it follows from the fact that $\chi(m) = 0$ and

$$\sum_{a \pmod q} \chi(a) e(am/q) = \sum_{y \pmod{q/(q,m)}} e(y) \sum_{x \pmod q} \chi(x) e(xq/d + y) = 0.$$

2. DUE ON 10TH OCTOBER

2.1. **Exercise.** Prove that $\Gamma(q)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and has index in it $q^3 \prod_{p|q} (1 - \frac{1}{p^2})$.

2.2. **Solution.** We consider the mod q reduction map

$$\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}),$$

whose kernel is by definition $\Gamma(q)$. Thus $\Gamma(q)$ is normal. Hence, as the above map is surjective, by the first isomorphism theorem

$$\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(q),$$

and so,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)] = |\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})|.$$

To compute the cardinality we first note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ then $(c, d, q) = 1$. For each such lower row (c, d) we have exactly q solutions for the congruence $ad - bc \equiv 1 \pmod{q}$. Thus the cardinality is,

$$q|\{(c, d) \pmod{q} \mid (c, d, q) = 1\}| = q \sum_{r|q} \mu(r)(q/r)^2 = q^3 \prod_{p|q} (1 - p^{-2}).$$

2.3. **Exercise.** Recall the subgroups $\Gamma_0(q)$, $\Gamma_1(q)$ and $\Gamma_d(q)$ of $\mathrm{SL}_2(\mathbb{Z})$ from the lectures. Compute indices of the subgroups in $\mathrm{SL}_2(\mathbb{Z})$.

2.4. **Solution.** Consider the surjective map

$$\Gamma_1(q) \rightarrow \mathbb{Z}/q\mathbb{Z},$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \pmod{q}.$$

The kernel of this map is by definition $\Gamma(q)$. Thus by the first isomorphism theorem,

$$\Gamma_1(q)/\Gamma(q) \cong \mathbb{Z}/q\mathbb{Z}.$$

Hence,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(q)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)][\Gamma_1(q) : \Gamma(q)]^{-1} = q^2 \prod_{p|q} (1 - p^{-2}).$$

Similarly, considering the map

$$\Gamma_0(q) \rightarrow (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod{q},$$

we conclude that

$$\Gamma_0(q)/\Gamma_1(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(q)] = \frac{1}{\phi(q)} q^2 \prod_{p|q} (1 - p^{-2}) = q \prod_{p|q} (1 + p^{-1}).$$

Again similarly, considering the map

$$\Gamma_d(q) \rightarrow (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \pmod{q},$$

we conclude that

$$\Gamma_d(q)/\Gamma(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_d(q)] \frac{1}{\phi(q)} q^3 \prod_{p|q} (1 - p^{-2}) = q^2 \prod_{p|q} (1 + p^{-1}).$$

2.5. Exercise. Prove that for any finite abelian group G one has $G \cong \hat{G}$.

Hint: First try to show for finite abelian groups G_1 and G_2 that $\hat{G}_1 \times \hat{G}_2 \cong \widehat{G_1 \times G_2}$. Then use the structure theory of the finite abelian groups.

2.6. Solution. We define a map

$$\hat{G}_1 \times \hat{G}_2 \rightarrow \widehat{G_1 \times G_2} \text{ by } (\chi_1, \chi_2) \mapsto \{\chi : (g_1, g_2) \mapsto \chi_1(g_1)\chi_2(g_2)\}.$$

This map is clearly well-defined homomorphism. To see injectivity if χ is the trivial character then

$$\chi_1(g_1) = \chi_2^{-1}(g_2) \forall (g_1, g_2) \in G_1 \times G_2,$$

which implies that χ_i are the trivial character. From the lecture we recall that $|G| = |\hat{G}|$, which proves the isomorphism. Now from the structure theory of the finite abelian groups we know that every finite abelian group is isomorphic to direct product of $\mathbb{Z}/n\mathbb{Z}$. hence it is enough to show that

$$\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \cong \mu_n \cong \mathbb{Z}/n\mathbb{Z},$$

where μ_n is the group of n 'th roots of unity. To See this isomorphism we consider that map

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \rightarrow \mu_n \text{ by } \chi \mapsto \chi(1).$$

This map is clearly a well-defined homomorphism, as $\chi(1)^n = \chi(n) = \chi(0) = 1$, i.e. $\chi(1) \in \mu_n$. If $\chi(1) = 1$ then $\chi(m) = \chi^m(1) = 1$, which proves the injectivity. Equality of the cardinalities concludes the proof.

2.7. **Exercise.** Recall the the product expansion

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(1) Use the above formula to prove that,

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left[\frac{1}{z-d} + \frac{1}{z+d} \right] = \pi \cot(\pi z) = \pi i - 2\pi i \sum_{d=0}^{\infty} e(dz).$$

(2) Prove that for even natural number k

$$\zeta(k) = -\frac{(2\pi i)^k}{2k!} B_k,$$

where B_k are the Bernoulli numbers.

(3) Prove that $\zeta(s)$ has zeros at negative even integers.

Hint: Use the functional equation of $\zeta(s)$.

2.8. **Solution.**

(1) We do a logarithmic differentiation of the given expression.

$$\begin{aligned} \pi \cot(\pi z) &= \frac{d}{dz} \log \sin(\pi z) \\ &= \frac{d}{dz} \log(\pi z) + \frac{d}{dz} \sum_{n=1}^{\infty} \log(1 - z^2/n^2) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2}, \end{aligned}$$

hence the first equality. For the second equality we see that,

$$\pi \cot(\pi z) = \pi i \frac{e(z) + 1}{e(z) - 1} = \pi i - 2\pi i \frac{1}{1 - e(z)} = \pi i - 2\pi i \sum_{n=0}^{\infty} e(nz),$$

completing the proof.

(2) Recall that the Bernoulli numbers are defined by the coefficient of the series expansion of $\frac{x}{e^x - 1}$, i.e.

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.$$

Consider the generating series of $\zeta(2k)$

$$1 + 2 \sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

For $|z| < 1$ the above sum is absolutely convergent, so plugging in the definition of $\zeta(s)$ for $s > 1$ and changing the order of the summation we get that above sum is

$$1 + 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (z/n)^{2k} = 1 + \sum_{n=1}^{\infty} \frac{2z^2}{n^2 - z^2} = \pi z \cot(\pi z),$$

where the last equality is from (1). But from (2)

$$\pi z \cot(\pi z) = \pi iz - \frac{2\pi iz}{1 - e^{2\pi iz}} = \pi iz - \sum_{k=0}^{\infty} B_k \frac{(2\pi iz)^k}{k!}.$$

Equating two power series we conclude that

$$2\zeta(2k) = -B_{2k} \frac{(2\pi i)^{2k}}{(2k)!},$$

concluding the result.

(3) We recall the functional equation of $\zeta(s)$

$$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \zeta(1-s)\pi^{(1-s)/2}\Gamma((1-s)/2).$$

We also recall the duplication formula,

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma(s/2)\Gamma((1+s)/2),$$

and

$$\Gamma(1/2 - s/2)\Gamma(1/2 + s/2) = \frac{\pi}{\cos(\pi s/2)}.$$

Combining all of them we get that,

$$\zeta(1-s) = 2(2\pi)^{-s} \cos(\pi s/2)\Gamma(s)\zeta(s).$$

Plugging in $s = 2n + 1$ for $n \geq 1$ and checking that $\cos(n\pi + \pi/2) = 0$ we conclude that

$$\zeta(-2n) = 0.$$

2.9. Eisenstein Series of weight 2. In the lecture we have defined Eisenstein series E_k of weight k for $k > 2$. In this exercise we will define *Eisenstein series* E_2 of weight 2 and will show that it satisfies an “almost modularity” relation.

2.10. Exercise. Define the following functions for $z \in \mathbb{H}$:

$$\begin{aligned} G_2(z) &:= \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2}, \\ G_2^*(z) &:= G_2(z) - \frac{\pi}{2\Im(z)}, \\ G_{2,\epsilon} &:= \frac{1}{2} \sum_{(m,n) \neq (0,0)} \frac{1}{(mz + n)^2} \frac{1}{|mz + n|^{2\epsilon}}, \text{ for } \epsilon > 0. \end{aligned}$$

- (1) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove that $G_{2,\epsilon}$ converges absolutely and locally uniformly. Also show that,

$$G_{2,\epsilon}(\gamma z) = (cz + d)^2 |cz + d|^{2\epsilon} G_{2,\epsilon}(z).$$

- (2) For $\epsilon > -1/2$ define:

$$I_\epsilon(z) := \int_{\mathbb{R}} \frac{dt}{(z+t)^2 |z+t|^{2\epsilon}} \text{ and } I(\epsilon) := \int_{\mathbb{R}} \frac{dt}{(i+t)^2 (1+t^2)^\epsilon}.$$

Consider

$$G_{2,\epsilon}(z) - \sum_{m=1}^{\infty} I_\epsilon(mz).$$

Use the mean value theorem to prove that it converges absolutely and locally uniformly for $\epsilon > -1/2$ and the limit as $\epsilon \rightarrow 0$ is $G_2(z)$.

- (3) Show that

$$I_\epsilon(z) = \frac{I(\epsilon)}{\Im(z)^{1+2\epsilon}} \text{ and } I'(0) = -\pi.$$

Use this to show that the limit of $G_{2,\epsilon}(z)$ as $\epsilon \rightarrow 0$ is $G_2^*(z)$. Hence G_2^* transforms like a modular form of weight 2.

- (4) Conclude that

$$G_2(\gamma z) = (cz + d)^2 G_2(z) - \pi ic(cz + d).$$

E_2 is defined to be, as usual, $\frac{G_2}{\zeta(2)}$.

2.11. Solution.

- (1) Note that, for $k > 2$ and $z \in \mathbb{H}$

$$\sum_{N=1}^{\infty} \sum_{N < |mz+n| \leq N+1} \frac{1}{|mz+n|^k} \leq \sum_{N=1}^{\infty} \frac{\#\{(m,n) \in \mathbb{Z}^2 \mid N \leq |mz+n| \leq N+1\}}{N^k}.$$

It is easy to check that

$$\#\{(m,n) \mid N \leq |mz+n| \leq N+1\} \ll \pi(N+1)^2 - \pi N^2 \ll N.$$

Thus the above sum is, as $k > 2$

$$\ll \sum_{N=1}^{\infty} N^{1-k} < \infty.$$

Now we see that,

$$G_{2,\epsilon} \leq \sum_{0 \leq |mz+n| \leq 1} |mz+n|^{-2-2\epsilon} + \sum_{1 \leq |mz+n|} |mz+n|^{-2-2\epsilon}.$$

The first sum has finite number of summands and second sum is absolutely and locally uniformly convergent by the previous argument. Thus the sum of $G_{2,\epsilon}$ are

convergent absolutely and locally uniformly, thus defines a holomorphic function on \mathbb{H} . To see the transformation law we first note that every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ induces a bijection from $\mathbb{Z}^2 \setminus \{(0,0)\}$ to itself by right multiplication. Also one checks that,

$$m\gamma z + n = \frac{(ma + nc)z + (mb + nd)}{cz + d} = \frac{m'z + n'}{cz + d}.$$

Combining these two facts, we conclude that

$$G_{2,\epsilon}(\gamma z) = \sum_{(m',n') \neq (0,0)} \frac{(cz + d)^2 |cz + d|^{2\epsilon}}{(m'z + n') |m'z + n'|^{2\epsilon}} = (cz + d)^2 |cz + d|^{2\epsilon} G_{2,\epsilon}(z).$$

(2) Let

$$f(t) := (mz + t)^2 |mz + t|^{-2\epsilon},$$

with implicit dependence on mz . Now as we have proved the absolute convergence of the $\sum f(n)$ we will freely change the order of summations and order of integration and summation, as follows.

$$\begin{aligned} \tilde{G}_{2,\epsilon}(z) &= G_{2,\epsilon}(z) - \sum_{m=0}^{\infty} I_{\epsilon}(mz) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} (f(n) - \int_n^{n+1} f(t) dt) \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \int_n^{n+1} (f(n) - f(t)) dt. \end{aligned}$$

By the mean value theorem on $n \leq t \leq n+1$ we get that

$$|f(n) - f(t)| \leq \sup_{n \leq u \leq n+1} |f'(u)| \ll |mz + n|^{-3-2\epsilon}.$$

Hence, the sum is absolutely convergent for $\epsilon > -1/2$ and thus $\lim_{\epsilon \rightarrow 0} \tilde{G}_{2,\epsilon}$ exists and defines a holomorphic function. We calculate,

$$\begin{aligned} &\lim_{\epsilon \rightarrow 0} \tilde{G}_{2,\epsilon}(z) \\ &= \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m=1}^{\infty} \left[\sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} + \sum_{n \in \mathbb{Z}} \left(\frac{1}{mz + n + 1} - \frac{1}{mz + n} \right) \right] \\ &= \frac{1}{2} \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} \\ &= G_2(z) \end{aligned}$$

(3) Let $z = x + iy$. Then changing variable $t \mapsto yt - x$ we get that,

$$\begin{aligned} I_\epsilon(x + iy) &= \int_{\mathbb{R}} \frac{dt}{(x + t + iy)^2 |x + t + iy|^{2\epsilon}} \\ &= \frac{1}{y^{1+2\epsilon}} \int_{\mathbb{R}} \frac{dt}{(t + i)^2 |t + i|^{2\epsilon}} = \frac{I(\epsilon)}{y^{1+2\epsilon}}. \end{aligned}$$

Differentiating under the integration sign and then integrating by parts we get that,

$$\begin{aligned} I'(0) &= - \int_{\mathbb{R}} \frac{\log(1 + t^2)}{(t + i)^2} dt = \frac{\log(1 + t^2)}{t + i} \Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{2t dt}{(t + i)(1 + t^2)} \\ &= - \int_{\mathbb{R}} \frac{1}{(t + i)^2} + \frac{1}{1 + t^2} = - \int_{\mathbb{R}} \frac{dt}{t^2 + 1} = -\pi. \end{aligned}$$

Using the above two results we compute that,

$$\lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} I_\epsilon(mz) = \lim_{\epsilon \rightarrow 0} \sum_{m=1}^{\infty} \frac{I(\epsilon)}{(my)^{1+2\epsilon}} = \lim_{\epsilon \rightarrow 0} \frac{I(\epsilon)\zeta(1 + 2\epsilon)}{\Im(z)^{1+2\epsilon}}.$$

From the exercise 1.6 we know that

$$\zeta(1 + 2\epsilon) = \frac{1}{2\epsilon} + O(1).$$

Using that $I(0) = 0$ we have that above limit equals to

$$\lim_{\epsilon \rightarrow 0} \frac{I(\epsilon)}{2\epsilon\Im(z)^{1+2\epsilon}} = \frac{I'(0)}{2\Im(z)}.$$

Thus,

$$\lim_{\epsilon \rightarrow 0} G_{2,\epsilon}(Z) = \lim_{\epsilon \rightarrow 0} \left(\tilde{G}_{2,\epsilon}(z) + \sum_{m=1}^{\infty} I_\epsilon(mz) \right) = G_2(z) - \frac{\pi}{2\Im(z)} = G_2^*(z).$$

(4) From part (1) and (3) letting $\epsilon \rightarrow 0$ we see that $G_2^*(z)$ transforms as a modular form of weight 2. So,

$$\begin{aligned} G_2(\gamma z) - (cz + d)^2 G_2(z) &= \frac{\pi}{2\Im(\gamma z)} - (cz + d)^2 \frac{\pi}{2\Im(z)} \\ &= \frac{\pi}{2\Im(z)} (|cz + d|^2 - (cz + d)^2) \\ &= \pi ic(cz + d), \end{aligned}$$

concluding the result.

3. DUE ON 24TH OCTOBER

3.1. **Exercise.** Prove the Bruhat decomposition: for any subfield $K \subset \mathbb{C}$

$$\mathrm{SL}_2(K) = N(K)A(K) \sqcup N(K)wN(K)A(K),$$

where the notations are same as in the lectures. Using this prove that the fractional linear transformation $\mathrm{GL}_2(\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C})$ preserves the lines.

3.2. **Exercise.** Recall the Fourier expansions of the Eisenstein series

$$E_k(z) = 1 + c_k \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where for $k = 2, 4, \dots, 14$ the c_k are $-24, 240, -504, 480, -264, 65520/691, -24$ with $q := e(z)$ and $\sigma_s(n) := \sum_{d|n} d^s$.

- (1) Use dimension formula to show that $E_8 = E_4^2$, $E_4E_6 = E_{10}$, and $E_6E_8 = E_{14}$. What relations can you get between σ_n 's using the above relations (some of them were obtained during the lectures)?
- (2) Define the *Serre derivative* by

$$D_k := \frac{1}{2\pi i} \frac{d}{dz} - \frac{k}{12} E_2.$$

Show that $D_k : M_k \rightarrow M_{k+2}$ and $Df \in S_{k+2}$ iff $f \in S_k$.

- (3) Calculate DE_4 and DE_6 . Find σ_5 in terms of σ_1 and σ_3 resp. and σ_7 in terms of σ_1 and σ_5 .

3.3. **Exercise.** Recall that the Delta function from the lecture defined in terms of some Eisenstein series. Here we start with a different definition and show equality afterwards.

$$\Delta(z) := q \prod_{n=1}^{\infty} (1 - q^n)^{24},$$

which has a Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n = q - 24q^2 + 252q^3 + O(q^4) \in \mathbb{Z}[[q]],$$

with $q = e(z)$ as usual. $\tau : \mathbb{N} \rightarrow \mathbb{C}$ is called *Ramanujan Tau function*.

- (1) Prove that $\frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) = (2\pi)^{-12} E_2(z)$ and conclude that $\Delta \in S_{12}$.
- (2) Show that $\Delta = \frac{E_4^3 - E_6^2}{1728}$, and derive τ in terms of σ_3 and σ_5 .
- (3) Show that $E_{12} - E_6^2 = c\Delta$ with $c = \frac{2^6 3^5 7^2}{691}$ and derive relation between τ , σ_{11} and σ_5 . Use this to prove the famous congruence by Ramanujan:

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691},$$

for all $n \geq 1$.

3.4. A Riemmanian metric on the upper half plane. A Riemmanian metric on \mathbb{H} can be defined as

$$ds^2(z) = \frac{d\Re^2(z) + d\Im^2(z)}{\Im^2(z)},$$

which gives \mathbb{H} a hyperbolic structure (More details in the upcoming lecture).

3.5. Exercise. Let $z_1, z_2 \in \mathbb{H}$. We define *geodesic segment between z_1 and z_2* to be the unique length minimizing curve (which exists) joining z_1 and z_2 under the hyperbolic metric as above. We define the hyperbolic distance between z_1 and z_2 to be

$$d_h(z_1, z_2) := \text{Length of geodesic segment between } z_1 \text{ and } z_2.$$

(1) Prove that

$$ds^2(gz) = ds^2(z), \quad \forall g \in \text{GL}_2^+(\mathbb{R}),$$

that is ds^2 is a $\text{GL}_2^+(\mathbb{R})$ invariant metric.

- (2) Prove that if $\Re(z_1) = \Re(z_2)$ then the geodesic segment joining them is the vertical line from joining z_1 and z_2 .
- (3) Prove that for general z_1 and z_2 the geodesic segment joining them is the arc of the unique half-circle centered on \mathbb{R} containing these two points.
- (4) Prove that

$$\cosh(d_h(z_1, z_2)) = 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)}.$$