MODULAR FORMS EXERCISES AND SOLUTIONS

1. Due on 26th September

1.1. **Exercise.** Let $\mathcal{P}$ be the set of primes. Prove that $\sum_{p\in\mathcal{P}} \frac{1}{p} = +\infty$.

1.2. **Summation by Parts.** Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, let $0 < y < x$ and let $f : [y, x] \to \mathbb{C}$ be a function with continuous derivative on $[y, x]$. Then

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where $A(x) = \sum_{n\leq x} a_n$.

1.3. **Exercise.** Prove that for every $\delta > 0$,

$$\pi(x) := |\{p \in \mathcal{P} \mid p \leq x\}|$$

is bigger than $\frac{x}{(\log x)^{1+\delta}}$ for some sufficiently large x.

1.4. **Exercise.** Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}dx,$$

where $\{x\}$ is the fractional part of x. Using this show that $\zeta(s)$ has analytic continuation to $\Re(s) > 0$ with a simple pole at $s = 1$.

1.5. **Exercise.** Prove that the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) := \int_0^\infty e^{-t}t^s\frac{dt}{t}$$

has analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple pole at each non-positive integer. Find the residues of the Gamma function at those poles.
Hint: First prove that $\Gamma(s + 1) = s\Gamma(s)$.

1.6. **Exercise.** Prove the Poisson summation formula: Let $f \in \mathcal{S}(\mathbb{R})$ be in the Schwartz class. Prove that

$$\sum_{n\in\mathbb{Z}} f(n + u) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e(nu).$$

Note: Putting $u = 0$ we get the usual Poisson summation formula.

1.7. **Exercise.** Recall that,

$$G(1,N) := \sum_{n \mod N} e(n^2/N).$$

Prove that

(1) For any odd positive integer N, $G(1,N^2) = \sqrt{N}$ and $G(N^3) = NG(N)$.
(2) For every positive integer N, $G(1,N) = \frac{1+i^{-N}}{1-i}\sqrt{N}$.

1.8. **Dirichlet Character.** A *Dirichlet character with modulus* q is a character

$$\chi : \mathbb{Z}/q\mathbb{Z}^\times \to \mathbb{C}^\times$$

extended to $\mathbb{Z}$ by making it q-periodic and defining $\chi(a) = 0$ for $(a,q) > 1$. Associated to each character χ, in addition to its modulus q, is a natural number q', its conductor. The *conductor* q' is the smallest divisor of q such that χ can be written as $\chi = \chi'\chi_0$, where χ_0 is the trivial Dirichlet character mod q and χ' is a character of modulus q'. If a character has conductor equal to to its modulus then it is called a *primitive Dirichlet character.* Check that, for a primitive Dirichlet character χ mod q one has

$$\frac{1}{q}\sum_{a \mod q} \chi(ma+b) = \begin{cases} \chi(b), & \text{if } q \mid m \\ 0, & \text{if } q \nmid m. \end{cases}$$

The above is not true for a non-primitive character.

1.9. **Exercise.** Let χ be a primitive Dirichlet character mod q and $f \in L^1(\mathbb{R})$. Prove that

$$\sum_{m \in \mathbb{Z}} f(m)\chi(m) = \frac{G(\chi)}{q}\sum_{n \in \mathbb{Z}} \hat{f}(n/q)\bar{\chi}(n),$$

where $G(\chi)$ is the Gauss sum attached to χ defined by

$$G(\chi) := \sum_{a \mod q} \chi(a)e(a/q).$$

Hint: Use the Poisson summation formula.