MODULAR FORMS EXERCISES AND SOLUTIONS

1. Due on 26th September

1.1. **Exercise.** Let $\mathcal{P}$ be the set of primes. Prove that $\sum_{p\in\mathcal{P}} \frac{1}{p} = +\infty$.

1.2. **Solution.** Let $s > 1$. Then from the Euler product of the Zeta function,

$$\begin{aligned}
\log \zeta(s) &= \sum_{p\in\mathcal{P}} -\log(1-p^{-s}) = \sum_{p\in\mathcal{P}} \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} \\
&\leq \sum_{p\in\mathcal{P}} \frac{1}{p^s} + \sum_{p\in\mathcal{P}} \sum_{k=2} \frac{1}{p^k} = \sum_{p\in\mathcal{P}} \frac{1}{p^s} + \sum_{p\in\mathcal{P}} \frac{1}{p(p-1)} \\
&= \sum_{p\in\mathcal{P}} \frac{1}{p^s} + O(1)
\end{aligned}$$

As we know that $\lim_{s\to 1+} \zeta(s) = +\infty$, letting $s \to 1+$ in the above inequality we conclude that

$$\lim_{s\to 1+} \sum_{p\in\mathcal{P}} \frac{1}{p^s} = +\infty,$$

hence the result.

1.3. **Summation by Parts.** Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, let $0 < y < x$ and let $f : [y, x] \to \mathbb{C}$ be a function with continuous derivative on $[y, x]$. Then

$$\sum_{y<n\leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where $A(x) = \sum_{n\leq x} a_n$.

1.4. **Exercise.** Prove that for every $\delta > 0$,

$$\pi(x) := |\{p \in \mathcal{P} \mid p \leq x\}|$$

is bigger than $\frac{x}{(\log x)^{1+\delta}}$ for some sufficiently large x.

1.5. **Solution.** Let a_n be the prime indicator function, i.e.

$$a_n := \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not a prime.} \end{cases}.$$

Using summation by parts we note that,

$$\sum_{p \leq x} \frac{1}{p} = \sum_{3/2 < n \leq x} \frac{a_n}{n} = \frac{\pi(x)}{x} + \int_{3/2}^{x} \frac{\pi(t)}{t^2} dt.$$

If the claim is false i.e. for all sufficiently large x, $\pi(x) \leq x/(\log x)^{1+\delta}$ then from the above,

$$\sum_{p \leq x} \frac{1}{p} \leq \frac{1}{(\log x)^{1+\delta}} + C + \frac{1}{(\log x)^{\delta}},$$

for some constant C. The RHS of the above tends to C as $x \to \infty$ contradicting Exercise 1.1, hence the result.

1.6. **Exercise.** Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\}$ is the fractional part of x. Using this show that $\zeta(s)$ has meromorphic continuation to $\Re(s) > 0$ with a simple pole at $s = 1$.

1.7. **Solution.** Let $\Re(s) > 1$. Then using the summation by parts as following.

$$\sum_{n \leq x} \frac{1}{n^s} = \frac{[x]}{x^s} + s \int_1^x \frac{[t]}{t^{s+1}} dt = \frac{1}{x^{s-1}} - \frac{\{x\}}{x^s} + s \int_1^x t^{-s} dt - s \int_1^x \frac{\{t\}}{t^{s+1}} dt$$

$$= \frac{s}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}} dt + O(x^{-\Re(s)} + x^{-\Re(s)+1}).$$

Letting $x \to \infty$, as $\Re(s) > 1$, we conclude that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$

We now note that the integral right hand side is well defined for $\Re(s) > 0$ and is holomorphic in s. As $\frac{s}{s-1}$ is a meromorphic function with simple pole at $s = 1$ and residue 1, we conclude the meromorphic continuation of $\zeta(s)$ to $\Re(s) > 0$.

1.8. **Exercise.** Prove that the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) := \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

has analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple pole at each non-positive integer. Find the residues of the Gamma function at those poles.
Hint: First prove that $\Gamma(s+1) = s\Gamma(s)$.

1.9. **Solution.** By integration by parts we see that

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1} \frac{dt}{t} = \int_0^\infty e^{-t} s t^s \frac{dt}{t} = s\Gamma(s),$$

for $\Re(s) > 0$. Thus $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ extends definition of $\Gamma(s)$ to $\Re(s) > -1$ meromorphically with pole at $s = 0$ as

$$\lim_{s\to 0+} \int_0^\infty e^{-t} t^s \frac{dt}{t} = +\infty.$$

The pole is simple, as $\lim_{s\to 0} s\Gamma(s) = 1$, and with residue 1. Similarly $\Gamma(s)$ can be extended to all $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple poles at $s = -n$, $n \in \mathbb{N}$ with residue,

$$\lim_{s\to -n} (s+n)\Gamma(s) = \lim_{s\to -n} \frac{\Gamma(s+n+1)}{(s+n-1)\dots s} = \frac{(-1)^n}{n!}.$$

1.10. **Exercise.** Prove the Poisson summation formula: Let $f \in \mathcal{S}(\mathbb{R})$ be in the Schwartz class. Prove that

$$\sum_{n\in\mathbb{Z}} f(n+u) = \sum_{n\in\mathbb{Z}} \hat{f}(n) e(nu).$$

Note: Putting $u = 0$ we get the usual Poisson summation formula.

1.11. **Solution.** Let

$$F(x) : \sum_{n\in\mathbb{Z}} f(n+x)$$

which is a function on $L^1(\mathbb{R}/\mathbb{Z})$ so has a Fourier expansion of the form

$$F(x) = \sum_{n\in\mathbb{Z}} e(nx)\hat{F}(n).$$

Here

$$\hat{F}(n) = \int_0^1 F(x)e(-nx)dx = \sum_{n\in\mathbb{Z}} \int_0^1 \sum_{m\in\mathbb{Z}} f(m+x)e(-nx)dx$$
$$= \sum_{m\in\mathbb{Z}} \int_m^{m+1} f(x)e(-nx) = \int_{-\infty}^\infty f(x)e(-nx)dx = \hat{f}(n),$$

this provides the result.

1.12. **Exercise.** Recall that,

$$G(1,N) := \sum_{n \mod N} e(n^2/N).$$

Prove that

(1) For any odd positive integer N, $G(1,N^2) = N$ and $G(1,N^3) = NG(1,N)$.
(2) For every positive integer N, $G(1,N) = \frac{1+i^{-N}}{1-i}\sqrt{N}$.

1.13. **Solution.** (1) is elementary. We can parametrize the residue class of N^k by

$$\{a_1 N^{k-1} + a_2 N^{k-2} + \cdots + a_k \mid 0 \le a_i \le N-1\}.$$

Using this we have,

$$\begin{aligned} G(1, N^2) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(aN+b)^2}{N^2}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \sum_{a=0}^{N-1} e\left(\frac{2ab}{N}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \delta_{b=0} N = N. \end{aligned}$$

Similarly,

$$\begin{aligned} G(1, N^3) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{c=0}^{N-1} e\left(\frac{(aN^2+bN+c)^2}{N^3}\right) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) \sum_{a=0}^{N-1} e(2ac/N) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) N\delta_{c=0} = NG(1, N). \end{aligned}$$

For the second part we use the Poisson summation formula. First we note the function

$$f(x) := 1_{[0,N]} e(x^2/N)$$

is a function which is continuous on $(0, N)$ and has continuity only from one side at $x = 0, N$. From the Fourier theory we know that the Fourier series of f at $x = 0$ would converge to $\frac{f(0+)+f(0-)}{2} = f(0+)/2$. and similarly, at $x = N$ to $f(N-)/2$ Thus using the (modified) Poisson summation formula and using that $f(0+) = f(N-)$ we get that,

$$\begin{aligned} \sum_{n=0}^{N} e(N^2/N) &= \frac{f(0+)}{2} + \sum_{n=1}^{N-1} f(n) + \frac{f(N-)}{2} \\ &= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(nx) dx = \sum_{n\in\mathbb{Z}} \int_0^N e(x^2/N + nx) dx. \end{aligned}$$

Thus,

$$G(1, N) = N \sum_{n\in\mathbb{Z}} \int_0^1 e(Nx^2 + nNx) dx = N \sum_{n\in\mathbb{Z}} e(-Nn^2/4) \int_0^1 e(N(x + n/2)^2)$$

Noting that

$$e(-Nn^2/4) = \begin{cases} 1, \text{ if } n \text{ is even,} \\ i^{-N}, \text{ if } n \text{ is odd.} \end{cases},$$

and dividing the above sum into odd and even parts we get that,

$$\begin{aligned} G(1,N) &= N \sum_{n\in\mathbb{Z}} \int_n^{1+n} e(Nx^2)dx + Ni^{-N} \sum_{n\in\mathbb{Z}} \int_{n-1/2}^{n+1/2} e(nx^2)dx \\ &= \sqrt{N}(1+i^{-N}) \int_{-\infty}^{\infty} e(y^2)dy. \end{aligned}$$

The last integral can be checked convergent and we call it C. Thus,

$$G(1,N) = \sqrt{N}C(1+i^{-N}).$$

Checking that, $G(1,1) = 1$, we conclude the result.

1.14. **Dirichlet Character.** A *Dirichlet character with modulus* q is a character

$$\chi : \mathbb{Z}/q\mathbb{Z}^\times \to \mathbb{C}^\times$$

extended to $\mathbb{Z}$ by making it q-periodic and defining $\chi(a) = 0$ for $(a,q) > 1$. Associated to each character χ, in addition to its modulus q, is a natural number q', its conductor. The *conductor* q' is the smallest divisor of q such that χ can be written as $\chi = \chi'\chi_0$, where χ_0 is the trivial Dirichlet character mod q and χ' is a character of modulus q'. If a character has conductor equal to to its modulus then it is called a *primitive Dirichlet character.* Check that, for a primitive Dirichlet character χ mod q one has

$$\frac{1}{q} \sum_{a \mod q} \chi(ma+b) = \begin{cases} \chi(b), \text{ if } q \mid m \\ 0, \text{ if } q \nmid m. \end{cases}$$

The above is not true for a non-primitive character.

1.15. **Exercise.** Let χ be a primitive Dirichlet character mod q and $f \in L^1(\mathbb{R})$. Prove that

$$\sum_{m\in\mathbb{Z}} f(m)\chi(m) = \frac{G(\chi)}{q} \sum_{n\in\mathbb{Z}} \hat{f}(n/q)\bar{\chi}(n),$$

where $G(\chi)$ is the Gauss sum attached to χ defined by

$$G(\chi) := \sum_{a \mod q} \chi(a)e(a/q).$$

Hint: Use the Poisson summation formula.

1.16. **Solution.** First we prove the following. Let $v \in \mathbb{R}$ and $u \in \mathbb{R}^+$. Then using the Poisson summation formula,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} f(um+v) &= \sum_{m\in\mathbb{Z}} \int_{\infty}^{\infty} f(ux+v)e(-mx)dx \\ &= \sum_{m\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e(-m(x-v)/u)\frac{dx}{u} \\ &= \frac{1}{u}\sum_{m\in\mathbb{Z}} \hat{f}(m)e(mv/u).\end{aligned}$$

Using the above we get that,

$$\begin{aligned}\sum_{m\in\mathbb{Z}} f(m)\chi(m) &= \sum_{m\in\mathbb{Z}} \sum_{a \mod q} \chi(a)f(mq+a) \\ &= \sum_{a \mod q} \chi(a)\frac{1}{q}\sum_{m\in\mathbb{Z}} \hat{f}(m)e(ma/q) \\ &= \frac{G(\chi)}{q}\sum_{m\in\mathbb{Z}} \hat{f}(m)\bar{\chi}(m).\end{aligned}$$

Here in the last line we have used that for a primitive Dirichlet character χ,

$$\sum_{a \mod q} \chi(a)e(am/q) = \bar{\chi}(m)G(\chi).$$

This can be seen as follows. Let $(m,q)=1$. Then,

$$\bar{\chi}(m)G(\chi) = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(am^{-1})e(a/q) = \sum_{a\in(\mathbb{Z}/q\mathbb{Z})^\times} \chi(a)e(am/q).$$

If $(m,q)>1$ then it follows from the fact that $\chi(m)=0$ and

$$\sum_{a \mod q} \chi(a)e(am/q) = \sum_{y \mod q/(q,m)} e(ym/q) \sum_{x \mod q} \chi(xq/d+y) = 0.$$

2. Due on 10th October

2.1. **Exercise.** Prove that $\Gamma(q)$ is a normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$ and has index in it $q^3\prod_{p|q}(1-\frac{1}{p^2})$.

2.2. **Solution.** We consider the $\mod q$ reduction map

$$\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}),$$

whose kernel is by definition $\Gamma(q)$. Thus $\Gamma(q)$ is normal. Hence, as the above map is surjective, by the first isomorphism theorem

$$\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z})/\Gamma(q),$$

and so,

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q)] = |\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})|.$$

To compute the cardinality we first note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})$ then $(c, d, q) = 1$. For each such lower row (c, d) we have exactly q solutions for the congruence $ad - bc \equiv 1 \mod q$. Thus the cardinality is,

$$q|\{(c, d) \mod q \mid (c, d, q) = 1\}| = q\sum_{r|q}\mu(r)(q/r)^2 = q^3\prod_{p|q}(1 - p^{-2}).$$

2.3. **Exercise.** Recall the subgroups $\Gamma_0(q)$, $\Gamma_1(q)$ and $\Gamma_d(q)$ of $\mathrm{SL}_2(\mathbb{Z})$ from the lectures. Compute indices of the subgroups in $\mathrm{SL}_2(\mathbb{Z})$.

2.4. **Solution.** Consider the surjective map

$$\Gamma_1(q) \to \mathbb{Z}/q\mathbb{Z},$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto b \mod q.$$

The kernel of this map is by definition $\Gamma(q)$. Thus by the first isomorphism theorem,

$$\Gamma_1(q)/\Gamma(q) \cong \mathbb{Z}/q\mathbb{Z}.$$

Hence,

$$[\mathrm{SL}_2(Z) : \Gamma_1(q)] = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma(q][\Gamma_1(q) : \Gamma(q)]^{-1} = q^2\prod_{p|q}(1 - p^{-2}).$$

Similarly, considering the map

$$\Gamma_0(q) \to (\mathbb{Z}/q\mathbb{Z})^\times,$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod q,$$

we conclude that

$$\Gamma_0(q)/\Gamma_1(q) \cong (\mathbb{Z}/q\mathbb{Z})^\times.$$

Thus,

$$[\mathrm{SL}_2(Z):\Gamma_1(q)] = \frac{1}{\phi(q)} q^2 \prod_{p|q}(1-p^{-2}) = q\prod_{p|q}(1+p^{-1}).$$

Again similarly, considering the map

$$\Gamma_d(q) \to (\mathbb{Z}/q\mathbb{Z})^{\times},$$

by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \mod q,$$

we conclude that

$$\Gamma_d(q)/\Gamma(q) \cong (\mathbb{Z}/q\mathbb{Z})^{\times}.$$

Thus,

$$[\mathrm{SL}_2(Z):\Gamma_d(q)]\frac{1}{\phi(q)} q^3 \prod_{p|q}(1-p^{-2}) = q^2\prod_{p|q}(1+p^{-1}).$$

2.5. **Exercise.** Prove that for any finite abelian group G one has $G \cong \hat{G}$.
Hint: First try to show for finite abelian groups G_1 and G_2 that $\hat{G}_1 \times \hat{G}_2 \cong \widehat{G_1 \times G_2}$. Then use the structure theory of the finite abelian groups.

2.6. **Solution.** We define a map

$$\hat{G}_1 \times \hat{G}_2 \to \widehat{G_1 \times G_2} \text{ by } (\chi_1,\chi_2) \mapsto \{\chi : (g_1,g_2) \mapsto \chi_1(g_1)\chi_2(g_2)\}.$$

This map is clearly well-defined homomorphism. To see injectivity if χ is the trivial character then

$$\chi_1(g_1) = \chi_2^{-1}(g_2) \forall (g_1,g_2) \in G_1 \times G_2,$$

which implies that χ_i are the trivial character. From the lecture we recall that $|G|= |\hat{G}|$, which proves the isomorphism. Now from the structure theory of the finite abelian groups we know that every finite abelian group is isomorphic to direct product of $\mathbb{Z}/_n\mathbb{Z}$. hence it is enough to show that

$$\widehat{\mathbb{Z}/n\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \cong \mu_n \cong \mathbb{Z}/n\mathbb{Z},$$

where μ_n is the group of n'th roots of unity. To See this isomorphism we consider that map

$$\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, S^1) \to \mu_n \text{ by } \chi \mapsto \chi(1).$$

This map is clearly a well-defined homomorphism, as $\chi(1)^n = \chi(n) = \chi(0) = 1$, i.e. $\chi(1) \in \mu_n$. If $\chi(1) = 1$ then $\chi(m) = \chi^m(1) = 1$, which proves the injectivity. Equality of the cardinalities concludes the proof.

2.7. **Exercise.** Recall the the product expansion

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

(1) Use the above formula to prove that,

$$\frac{1}{z} + \sum_{d=1}^{\infty} \left[\frac{1}{z-d} + \frac{1}{z+d}\right] = \pi \cot(\pi z) = \pi i - 2\pi i \sum_{d=0}^{\infty} e(dz).$$

(2) Prove that for even natural number k

$$\zeta(k) = -\frac{(2\pi i)^k}{2k!} B_k,$$

where B_k are the Bernoulli numbers.

(3) Prove that $\zeta(s)$ has zeros at negative even integers.
Hint: Use the functional equation of $\zeta(s)$.

2.8. **Solution.**

(1) We do a logarithmic differentiation of the given expression.

$$\begin{aligned}\pi \cot(\pi z) &= \frac{d}{dz} \log \sin(\pi z) \\ &= \frac{d}{dz} \log(\pi z) + \frac{d}{dz} \sum_{n=1}^{\infty} \log(1 - z^2/n^2) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2},\end{aligned}$$

hence the first equality. For the second equality we see that,

$$\pi \cot(\pi z) = \pi i \frac{e(z)+1}{e(z)-1} = \pi i - 2\pi i \frac{1}{1-e(z)} = \pi i - 2\pi i \sum_{n=0}^{\infty} e(nz),$$

completing the proof.

(2) Recall that the Bernoulli numbers are defined by the coefficient of the series expansion of $\frac{x}{e^x - 1}$, i.e.

$$\sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = \frac{x}{e^x - 1}.$$

Consider the generating series of $\zeta(2k)$

$$1 + 2\sum_{k=1}^{\infty} \zeta(2k) z^{2k}.$$

For $|z|< 1$ the above sum is absolutely convergent, so plugging in the definition of $\zeta(s)$ for $s > 1$ and changing the order of the summation we get that above sum is

$$1+2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(z/n)^{2k} = 1+\sum_{n=1}^{\infty}\frac{2z^2}{n^2-z^2} = \pi z\cot(\pi z),$$

where the last equality is from (1). But from (2)

$$\pi z\cot(\pi z) = \pi iz - \frac{2\pi iz}{1-e^{2\pi iz}} = \pi iz - \sum_{k=0}^{\infty} B_k\frac{(2\pi iz)^k}{k!}.$$

Equating two power series we conclude that

$$2\zeta(2k) = -B_{2k}\frac{(2\pi i)^{2k}}{(2k)!},$$

concluding the result.

(3) We recall the functional equation of $\zeta(s)$

$$\zeta(s)\pi^{-s/2}\Gamma(s/2) = \zeta(1-s)\pi^{(1-s)/2}\Gamma((1-s)/2).$$

We also recall the duplication formula,

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma(s/2)\Gamma((1+s)/2),$$

and

$$\Gamma(1/2-s/2)\Gamma(1/2+s/2) = \frac{\pi}{\cos(\pi s/2)}.$$

Combining all of them we get that,

$$\zeta(1-s) = 2(2\pi)^{-s}\cos(\pi s/2)\Gamma(s)\zeta(s).$$

Plugging in $s = 2n+1$ for $n \geq 1$ and checking that $\cos(n\pi+\pi/2) = 0$ we conclude that

$$\zeta(-2n) = 0.$$

2.9. **Eisenstein Series of weight 2.** In the lecture we have defined Eisenstein series E_k of weight k for $k > 2$. In this exercise we will define *Eisenstein series* E_2 *of weight* 2 and will show that it satisfies an "almost modularity" relation.

2.10. **Exercise.** Define the following functions for $z \in \mathbb{H}$:

$$G_2(z) := \sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2},$$

$$G_2^*(z) := G_2(z) - \frac{\pi}{2\Im(z)},$$

$$G_{2,\epsilon} := \frac{1}{2}\sum_{(m,n)\neq(0,0)}\frac{1}{(mz+n)^2}\frac{1}{|mz+n|^{2\epsilon}}, \text{ for } \epsilon > 0.$$

(1) Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Prove that $G_{2,\epsilon}$ converges absolutely and locally uniformly. Also show that,

$$G_{2,\epsilon}(\gamma z) = (cz+d)^2|cz+d|^{2\epsilon}G_{2,\epsilon}(z).$$

(2) For $\epsilon > -1/2$ define:

$$I_\epsilon(z) := \int_{\mathbb{R}} \frac{dt}{(z+t)^2|z+t|^{2\epsilon}} \text{ and } I(\epsilon) := \int_{\mathbb{R}} \frac{dt}{(i+t)^2(1+t^2)^\epsilon}.$$

Consider

$$G_{2,\epsilon}(z) - \sum_{m=1}^{\infty} I_\epsilon(mz).$$

Use the mean value theorem to prove that it converges absolutely and locally uniformly for $\epsilon > -1/2$ and the limit as $\epsilon \to 0$ is $G_2(z)$.

(3) Show that

$$I_\epsilon(z) = \frac{I(\epsilon)}{\Im(z)^{1+2\epsilon}} \text{ and } I'(0) = -\pi.$$

Use this to show that the limit of $G_{2,\epsilon}(z)$ as $\epsilon \to 0$ is $G_2^*(z)$. Hence G_2^* transforms like a modular form of weight 2.

(4) Conclude that

$$G_2(\gamma z) = (cz+d)^2 G_2(z) - \pi i c(cz+d).$$

E_2 is defined to be, as usual, $\frac{G_2}{\zeta(2)}$.

2.11. **Solution.**

(1) Note that, for $k > 2$ and $z \in \mathbb{H}$

$$\sum_{N=1}^{\infty} \sum_{N<|mz+n|\leq N+1} \frac{1}{|mz+n|^k} \leq \sum_{N=1}^{\infty} \frac{\#\{(m,n)\in\mathbb{Z}^2 \mid N \leq |mz+n| \leq N+1\}}{N^k}.$$

It is easy to check that

$$\#\{(m,n) \mid N \leq |mz+n| \leq N+1\} \ll \pi(N+1)^2 - \pi N^2 \ll N.$$

Thus the above sum is, as $k > 2$

$$\ll \sum_{N=1}^{\infty} N^{1-k} < \infty.$$

Now we see that,

$$G_{2,\epsilon} \leq \sum_{0\leq|mz+n|\leq 1} |mz+n|^{-2-2\epsilon} + \sum_{1\leq|mz+n|} |mz+n|^{-2-2\epsilon}.$$

The first sum has finite number of summands and second sum is absolutely and locally uniformly convergent by the previous argument. Thus the sum of $G_{2,\epsilon}$ are convergent abolustely and locally uniformly, thus defines a holomorphic function on $\mathbb{H}$. To see the transformation law we first note that every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ induces a bijection from $\mathbb{Z}^2 \setminus \{(0,0)\}$ to itself by right multiplication. Also one checks that,

$$m\gamma z + n = \frac{(ma+nc)z + (mb+nd)}{cz+d} = \frac{m'z+n'}{cz+d}.$$

Combining these two facts, we conclude that

$$G_2,\epsilon(\gamma z) = \sum_{(m',n')\neq(0,0)} \frac{(cz+d)^2|cz+d|^{2\epsilon}}{(m'z+n')|m'z+n'|^{2\epsilon}} = (cz+d)^2|cz+d|^{2\epsilon} G_{2,\epsilon}(z).$$

(2) Let

$$f(t) := (mz+t)^2|mz+t|^{-2\epsilon},$$

with implicit dependence on mz. Now as we have proved the absolute convergence of the $\sum f(n)$ we will freely change the order of summations and order of integration and summation, as follows.

$$\begin{aligned}
\tilde{G}_{2,\epsilon}(z) &= G_{2,\epsilon}(z) - \sum_{m=0}^{\infty} I_\epsilon(mz) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} (f(n) - \int_n^{n+1} f(t)dt) \\
&= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\epsilon}} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \int_n^{n+1} (f(n) - f(t))dt.
\end{aligned}$$

By the mean value theorem on $n \leq t \leq n+1$ we get that

$$|f(n) - f(t)| \leq \sup_{n\leq u\leq n+1} |f'(u)| \ll |mz+n|^{-3-2\epsilon}.$$

Hence, the sum is absolutely convergent for $\epsilon > -1/2$ and thus $\lim_{\epsilon\to 0} \tilde{G}_{2,\epsilon}$ exists and defines a holomorphic function. We calculate,

$$\begin{aligned}
&\lim_{\epsilon\to 0} \tilde{G}_{2,\epsilon}(z) \\
&= \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \sum_{m=1}^{\infty}\left[\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2} + \sum_{n\in\mathbb{Z}}\left(\frac{1}{mz+n+1} - \frac{1}{mz+n}\right)\right] \\
&= \frac{1}{2}\sum_{n\neq 0}\frac{1}{n^2} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2} \\
&= G_2(z)
\end{aligned}$$

(3) Let $z = x + iy$. Then changing variable $t \mapsto yt - x$ we get that,

$$\begin{aligned} I_\epsilon(x+iy) &= \int_{\mathbb{R}} \frac{dt}{(x+t+iy)^2|x+t+iy|^{2\epsilon}} \\ &= \frac{1}{y^{1+2\epsilon}} \int_{\mathbb{R}} \frac{dt}{(t+i)^2|t+i|^{2\epsilon}} = \frac{I(\epsilon)}{y^{1+2\epsilon}}. \end{aligned}$$

Differentiating under the integration sign and then integrating by parts we get that,

$$\begin{aligned} I'(0) &= -\int_{\mathbb{R}} \frac{\log(1+t^2)}{(t+i)^2} dt = \frac{\log(1+t^2)}{t+i}\Bigg|_{-\infty}^{\infty} - \int_{\mathbb{R}} \frac{2tdt}{(t+i)(1+t^2)} \\ &= -\int_{\mathbb{R}} \frac{1}{(t+i)^2} + \frac{1}{1+t^2} = -\int_{\mathbb{R}} \frac{dt}{t^2+1} = -\pi. \end{aligned}$$

Using the above two results we compute that,

$$\lim_{\epsilon \to 0} \sum_{m=1}^{\infty} I_\epsilon(mz) = \lim_{\epsilon \to 0} \sum_{m=1}^{\infty} \frac{I(\epsilon)}{(my)^{1+2\epsilon}} = \lim_{\epsilon \to 0} \frac{I(\epsilon)\zeta(1+2\epsilon)}{\Im(z)^{1+2\epsilon}}.$$

From the exercise 1.6 we know that

$$\zeta(1+2\epsilon) = \frac{1}{2\epsilon} + O(1).$$

Using that $I(0) = 0$ we have that above limit equals to

$$\lim_{\epsilon \to 0} \frac{I(\epsilon)}{2\epsilon \Im(z)^{1+2\epsilon}} = \frac{I'(0)}{2\Im(z)}.$$

Thus,

$$\lim_{\epsilon \to 0} G_{2,\epsilon}(Z) = \lim_{\epsilon \to 0} \left(\tilde{G}_{2,\epsilon}(z) + \sum_{m=1}^{\infty} I_\epsilon(mz) \right) = G_2(z) - \frac{\pi}{2\Im(z)} = G_2^*(z).$$

(4) From part (1) and (3) letting $\epsilon \to 0$ we see that $G_2^*(z)$ transforms as a modular form of weight 2. So,

$$\begin{aligned} G_2(\gamma z) - (cz+d)^2 G_2(z) &= \frac{\pi}{2\Im(\gamma z)} - (cz+d)^2 \frac{\pi}{2\Im(z)} \\ &= \frac{\pi}{2\Im(z)} (|cz+d|^2 - (cz+d)^2) \\ &= \pi i c(cz+d), \end{aligned}$$

concluding the result.

3. Due on 24th October

3.1. **Exercise.** Prove the Bruhat decomposition: for any subfield $K \subset \mathbb{C}$

$$\mathrm{SL}_2(K) = N(K)A(K) \sqcup N(K)wN(K)A(K),$$

where the notatons are same as in the lectures. Using this prove that the fractional linear transformation $\mathrm{GL}_2(\mathbb{C}) \curvearrowright \mathbb{P}^1(\mathbb{C})$ preserves the lines.

3.2. **Solution.** Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. if $c = 0$ then g is upper triangual so lies in NA. So let us assume that $c \neq 0$. So $b = ad/c$. Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} w \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1/c \end{pmatrix}.$$

This also can be proved in much more geometric way. First check that

$$g.\infty = a/c \implies \mathrm{Stab}_{\mathrm{GL}(2)}(\infty) = NA.$$

We prove that if $g \notin NA$ then $g \in NwNA$. To check this we see that

$$\begin{pmatrix} 1 & -a/c \\ & 1 \end{pmatrix} g.z = g.z - a/c = \frac{az+b}{cz+d} - \frac{a}{c} = \frac{1}{c^2z+cd} = w.c^2z + cd = w \begin{pmatrix} 1 & cd \\ & 1 \end{pmatrix} \begin{pmatrix} c & \\ & 1/c \end{pmatrix}.z.$$

To check that this decombosition is unique we note that, again, if $g = b \in NA$ this is obvious. If $g = nwb = n'wb'$ then

$$g.\infty = n.0 = n'.0 \implies n = n' \implies b = b'.$$

This proves the first part.
For the second part we first recall that a line in $\mathbb{P}^1(\mathbb{C})$ is of the form $L \cup \{\infty\}$ where L is a line or a circle in $\mathbb{C}$. As from the previous part and the fact that

$$\mathrm{GL}_2(\mathbb{C}) \cong Z(\mathbb{C})\mathrm{SL}_2(\mathbb{C}),$$

it is enough to prove that Z, N, A, w preserves the lines. While Z, N, A transforms in affine way, i.e.

$$z \mapsto az + b, \quad a \in \mathbb{C}^\times, b \in \mathbb{C}$$

it is clear that they preserve lines. Thus it is enough to check that w preserves a line L. Now, as we can freely move object in affine way, we may assume that L is a horizontal line passing through 0, i.e. $\Im(z) = 0$ or a unit circle centered at origin, i.e. $|z| = 1$. In either case the fact that

$$w.z = -\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

proves the claim.

3.3. **Exercise.** Recall the Fourier expansions of the Eisenstein series

$$E_k(z) = 1 + c_k \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where for $k = 2, 4, \ldots, 14$ the c_k are $-24, 240, -504, 480, -264, 65520/691, -24$ with $q := e(z)$ and $\sigma_s(n) := \sum_{d|n} d^s$.

(1) Use dimension formula to show that $E_8 = E_4^2$, $E_4E_6 = E_{10}$, and $E_6E_8 = E_{14}$. What relations can you get between σ_n's using the above relations (some of them were obtained during the lectures)?
(2) Define the *Serre derivative* by

$$D_k := \frac{1}{2\pi i}\frac{d}{dz} - \frac{k}{12}E_2.$$

Show that $D_k : M_k \to M_{k+2}$ and $D_k f \in S_{k+2}$ iff $f \in S_k$.
(3) Calculate DE_4 and DE_6. Find σ_5 in terms of σ_1 and σ_3 resp. and σ_7 in terms of σ_1 and σ_5.

3.4. **Solution.**

(1) Check that from the dimension formula that m_8, M_10, and M_14 are one dimensional. Therefore, $E^8 - cE_4^2$, $E_4E_6 = de_{10}$, and $E_6E_8 = eE_{14}$. But from the Fourier expansions of the Eisenstein series that their first Fourier coefficients are one we cocnlude that $c = d = e = 1$. Now multiplying the Fourier expansions we get that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m),$$

$$-11\sigma_9(n) = 10\sigma_3(n) - 21\sigma_5(n) - 5040\sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m),$$

$$-\sigma_{13}(n) = -21\sigma_5(n) + 20\sigma_7(n) - 10080\sum_{m=1}^{n-1} \sigma_5(m)\sigma_7(n-m).$$

(2) Let $f \in M_k$. As E_2, f, and f' are holomorphic so is $D_k f$. So it is enough to show that $D_k f$ transforms as a weight $k+2$ form to prove that image of D_k is in M_{k+2}. We check that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $j(\gamma, z) = cz + d$, and recalling from exercise 2.10(4) that

$$E_2(\gamma z) = j(\gamma, z)^2 E_2(z) + \frac{12cj(\gamma, z)}{2\pi i}.$$

we get that

$$\begin{aligned}
D_k f(\gamma z) &= \frac{1}{2\pi i} f'(\gamma z) - \frac{k}{12} E_2(\gamma z) f(\gamma z) \\
&= \frac{1}{2\pi i} j^2(\gamma, z) \frac{df(\gamma z)}{dz} - j^{k+2}(\gamma, z) E_2(z) f(z) - \frac{ckj^{k+1}(\gamma, z)}{2\pi i} f(z) \\
&= \frac{1}{2\pi i} j^2(\gamma, z) \frac{d}{dz} j^k(\gamma, z) f(z) - j^{k+2}(\gamma, z) E_2(z) f(z) - f(z) \frac{j^2(\gamma, z)}{2\pi i} \frac{d}{dz} j^k(\gamma, z) \\
&= \frac{j^{k+2}(\gamma, z)}{2\pi i} f'(z) - j^{k+2}(\gamma, z) E_2(z) f(z) \\
&= j^{k+2}(\gamma, z) D_k f(z).
\end{aligned}$$

Now note that,

$$q = e(z) \implies \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}.$$

Thus if f has Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n q^n,$$

then

$$D_k f = q \frac{df}{dq} - \frac{k}{12} E_2 f = \sum_{n=0}^{\infty} n a_n q^n + \frac{k}{12} E_2 f.$$

Thus it is clear that the zeroth Fourier coefficient is $-ka_0/12$ and that will be zero if and only if $a_0 = 0$ which proves the second claim.

(3) By part (2) $DE_4 \in M_6$ and $DE_6 \in M_8$. From the dimension formulas and the zeroth Fourier coefficients we conclude as in (1) that

$$DE_4 = cE_6, \quad c \in \mathbb{C},$$

with $c = -1/3$. Similarly, $DE_6 = -\frac{1}{2} E_8$. Now as in (1) comparing the Fourier coefficients we get that

$$21\sigma_5(n) = (30n - 10)\sigma_3(n) + \sigma_1(n) + 240 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_3(n-m),$$

$$20\sigma_7(n) = (42n - 21)\sigma_5(n) + \sigma_1(n) + 504 \sum_{m=1}^{n-1} \sigma_1(m)\sigma_5(n-m).$$

3.5. **Exercise.** Recall that the Delta function from the lecture defined in terms of some Eisenstein series. Here we start with a different defintion and show equality afterwards.

$$\Delta(z) := q\prod_{n=1}^{\infty}(1-q^n)^{24},$$

which has a Fourier expansion

$$\Delta(z) = \sum_{n=1}^{\infty}\tau(n)q^n = q - 24q^2 + 252q^3 + O(q^4) \in \mathbb{Z}[[q]],$$

with $q = e(z)$ as usual. $\tau : \mathbb{N} \to \mathbb{C}$ is called *Ramanujan Tau function.*

(1) Prove that $\frac{1}{2\pi i}\frac{d}{dz}\log\Delta(z) = E_2(z)$ and conclude that $\Delta \in S_{12}$.
(2) Show that $\Delta = \frac{E_4^3 - E_6^2}{1728}$, and derive τ in terms of σ_3 and σ_5.
(3) Show that $E_{12} - E_6^2 = c\Delta$ with $c = \frac{2^6 3^5 7^2}{691}$ and derive relation between τ, σ_{11} and σ_5. Use this to prove the famous congruence by Ramanujan:

$$\tau(n) \equiv \sigma_{11}(n) \mod 691,$$

for all $n \geq 1$.

3.6. **Solution.**

(1) Recall that $\frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$. Therefore,

$$\begin{aligned}
\frac{1}{2\pi i}\frac{d}{dz}\log\Delta(z) &= q\frac{d}{dq}\log\left(q\prod_{n=1}^{\infty}(1-q^n)^{24}\right) \\
&= q\frac{d}{dq}\left[\log q + 24\sum_{n=1}^{\infty}24\log(1-q^n)\right] \\
&= q\frac{d}{dq}\left[\log q - 24\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{q^{nk}}{k}\right] \\
&= 1 - 24\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}nq^{nk} \\
&= 1 - 24\sum_{n=1}^{\infty}q^n\left(\sum_{k|n}k\right) = E_2(z).
\end{aligned}$$

All interchanges of orders of summations are justified as the series is absolutely convergent as $|q|< 1$. Now from the product form it is clear that Δ is holomorphic and has zero as zeroth Fourier coefficient. So to prove that $\Delta \in S_{12}$ it is enough to show that Δ transforms as a weight 12 modular form. To check that keeping the same notations as in the solution 3.4(2) we compute that

$$\begin{aligned}
&\frac{1}{2\pi i}\frac{d}{dz}\log\Delta(\gamma z)\\
&= j(\gamma z)^{-2}\frac{1}{2\pi i}\frac{d}{dz}\log\Delta|_{\gamma z}\\
&= j(\gamma z)^{-2}E_2(\gamma z)\\
&= E_2(z)+\frac{12c}{2\pi i j(\gamma, z)}\\
&= \frac{1}{2\pi i}\frac{d}{dz}\log\Delta(z)+\frac{1}{2\pi i}\frac{d}{dz}\log j^{12}(\gamma, z)\\
&= \frac{1}{2\pi i}\frac{d}{dz}\log(j^{12}(\gamma, z)\Delta(z)).
\end{aligned}$$

Thus for each $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exists a constant $0 \neq c(\gamma)$ such that

$$\Delta(\gamma z) = c(\gamma) j^{12}(\gamma, z)\Delta(z).$$

It suffices to show that $c(\gamma) = 1$ for all γ. It is easy to check that

$$c : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^{\times}, \quad \gamma \mapsto c(\gamma)$$

a character. Thus it is enough to prove that $c(T) = 1$ and $c(S) = 1$ where T, S are the usual generators of $\mathrm{SL}_2(\mathbb{Z})$. But as Δ is 1-periodic so $c(T) = 1$. Now as $S.i = i$ and $\Delta(i) \neq 0$ we see that

$$c(S) = i^{-12} = 1,$$

completing the proof.

(2) As S_{12} is one dimensional and $E_4^3 - E_6^2$ has zero zeroth Fourier coefficient hence,

$$E_4^3 - E_6^2 = d\Delta, \quad d \in \mathbb{C}.$$

d can be calculated to be 1728 from the first Fourier coefficients of E_4 and E_6. Thus equating Fourier coefficients we conclude that

$$\begin{aligned}
12\tau(n) = 5\sigma_3(n) &+ 1200\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m) + 96000\sum_{r=1}^{n-1}\sum_{m=1}^{r-1}\sigma_3(m)\sigma_3(r-m)\sigma_3(n-r)\\
&+ 7\sigma_5(n) - 1764\sum_{m=1}^{n-1}\sigma_5(m)\sigma_5(n-m).
\end{aligned}$$

(3) Again by dimension formula arguing that S_{12} is one dimensional and comparing the first Fourier coefficients we conclude that

$$E_{12} - E_6^2 = \frac{2^6 3^5 7^2}{691}\Delta.$$

Comparing the Fourier coefficients we get that

$$2^6 3^5 7^2 \tau(n) = 65520\sigma_{11}(n) + 691.2.504\sigma_5(n) - 691.504^2 \sum_{m=1}^{n-1} \sigma_5(m)\sigma_5(n-m).$$

Dividing by 1008 and reducing mod 691 we conclude that

$$756\tau(n) \equiv 65\tau(n) \equiv 65\sigma_{11}(n) \mod 691.$$

As $(65, 691) = 1$ we conclude the final result.

3.7. **A Riemmanian metric on the upper half plane.** A Riemmanian metric on $\mathbb{H}$ can be defined as

$$ds^2(z) = \frac{d\Re^2(z) + d\Im^2(z)}{\Im^2(z)},$$

which gives $\mathbb{H}$ a hyperbolic structure (More details in the upcoming lecture).

3.8. **Exercise.** Let $z_1, z_2 \in \mathbb{H}$. We define *geodesic segment between* z_1 *and* z_2 to be the unique length minimizing curve (which exists) joining z_1 and z_2 under the hyperbolic metric as above. We define the hyperbolic distance between z_1 and z_2 to be

$$d_h(z_1, z_2) := \text{Length of geodesic segment between } z_1 \text{ and } z_2.$$

(1) Prove that
$$ds^2(gz) = ds^2(z), \quad \forall g \in \mathrm{GL}_2^+(\mathbb{R}),$$
that is ds^2 is a $\mathrm{GL}_2^+(\mathbb{R})$ invariant metric.

(2) Prove that if $\Re(z_1) = \Re(z_2)$ then the geodesic segment joining them is the vertical line joining z_1 and z_2.

(3) Prove that for general z_1 and z_2 the geodesic segment joining them is the arc of the unique half-circle centered on $\mathbb{R}$ containing these two points.

(4) Prove that
$$\cosh(d_h(z_1, z_2)) = 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)}.$$

3.9. **Solution.**

(1) Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then we check that

$$\frac{d(gz)}{dz} = \frac{\det(g)}{(cz+d)^2}.$$

Also recall that

$$\Im(gz) = \frac{\det(g)\Im(z)}{|cz+d|^2}.$$

Thus

$$ds^2(gz) = \frac{|d(gz)|^2}{\Im(gz)^2} = \frac{|\det(g)|^2}{|cz+d|^4}|dz|^2 \frac{|cz+d|^4}{|\det(g)|^2 \Im(z)^2} = \frac{|dz|^2}{\Im(z)^2} = ds^2(z).$$

(2) WLOG let $\Im(z_2) \geq \Im(z_1)$. Note that, the vertical path joining z_1 and z_2 can be given as

$$\phi(t) = \Re(z_1) + i\Im(z_1)\left(\frac{\Im(z_2)}{\Im(z_1)}\right)^t.$$

It is easy to check that the length of ϕ

$$L(\phi) = \log \Im(z_2) - \log \Im(z_1).$$

Let ϕ' be any other curve joining z_1 and z_2. Then the length of ϕ_1

$$L(\phi_1) = \int_0^1 \frac{|\phi_1'(t)|}{\Im(\phi_1(t))}dt \geq \int_0^1 \frac{\Im(\phi_1'(t))}{\Im(\phi_1(t))}dt = \log \Im(z_2) - \log \Im)z_1),$$

which proves the claim.

(3) First we claim that there exists a $g \in \mathrm{SL}_2(\mathbb{R})$ such that

$$\Re(gz_1) = \Re(gz_2) = 0.$$

First we assume the claim. Then we see that the length minimizing curve joining gz_1 and gz_2, them having same real part, is a vertical segment ϕ as in the previous part. As $\mathrm{SL}_2(\mathbb{R})$ acts by isometry the geodesic joining z_1 and z_2 would be $g^{-1}\phi$. From Exercise 3.1 we can conclude that $\mathrm{SL}_2(\mathbb{R})$ preserves lines in $\mathbb{P}^1(\mathbb{R}) \cong \mathbb{H} \cup \{\infty\}$, where lines in $\mathbb{P}^1(\mathbb{R})$ are vertical lines or half-circles centered in $\mathbb{R}$. This concludes the proof assuming the claim.

Now we turn to prove the claim. By transitivity property of $\mathrm{SL}_2(\mathbb{R})$ action one can find g such that $gz_1 = i$. Now as we know that $\mathrm{SO}(2)$ fixes i for any $k \in \mathrm{SO}(2)$ we have $gki = z_1$. So it is enough to find some k such that $\Re(kg^{-1}z_2) = 0$. For any $z \in \mathbb{H}$ we can always find $k \in \mathrm{SO}(2)$ such that $\Re(kz) = 0$. If $k = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $z = x + iy$) then to make sure that $\Re(kz) = 0$ one needs to see whether

$$\tan(2\theta) = -\frac{x}{y^2 + 1 - x^2},$$

which clearly exists.

(4) By the argument in the part (3) we can find $g \in \mathrm{SL}_2(\mathbb{R})$ such that gz_1 and gz_2 has zero real parts. Also from part (1) we know that g acts by isometry thus it is enough to prove the statement for z_1 and z_2 purely imaginary. But in part (2) we have proved that for such $z_i \in i\mathbb{R}$ one has

$$d_h(z_1, z_2) = |\log \Im(z_1) - \log \Im(z_2)| = |\log(z_1/z_2)|.$$

Thus,

$$\begin{aligned}\cosh(d_h(z_1, z_2)) &= \frac{1}{2}\left(e^{d_h(z_1,z_2)} + e^{-d_h(z_1,z_2)}\right) \\ &= \frac{1}{2}\left|\frac{z_1}{z_2} + \frac{z_1}{z_2}\right| 2 = \frac{|z_1^2 + z_2^2|}{2|z_1 z_2|} \\ &= 1 + \frac{|z_1 - z_2|^2}{2\Im(z_1)\Im(z_2)},\end{aligned}$$

completing the proof.