

MODULAR FORMS EXERCISES AND SOLUTIONS

1. DUE ON 26TH SEPTEMBER

1.1. **Exercise.** Let \mathcal{P} be the set of primes. Prove that $\sum_{p \in \mathcal{P}} \frac{1}{p} = +\infty$.

1.2. **Solution.** Let $s > 1$. Then from the Euler product of the Zeta function,

$$\begin{aligned} \log \zeta(s) &= \sum_{p \in \mathcal{P}} -\log(1 - p^{-s}) = \sum_{p \in \mathcal{P}} \sum_{k=1}^{\infty} \frac{1}{kp^{ks}} \\ &\leq \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \sum_{k=2}^{\infty} \frac{1}{p^k} = \sum_{p \in \mathcal{P}} \frac{1}{p^s} + \sum_{p \in \mathcal{P}} \frac{1}{p(p-1)} \\ &= \sum_{p \in \mathcal{P}} \frac{1}{p^s} + O(1) \end{aligned}$$

As we know that $\lim_{s \rightarrow 1+} \zeta(s) = +\infty$, letting $s \rightarrow 1+$ in the above inequality we conclude that

$$\lim_{s \rightarrow 1+} \sum_{p \in \mathcal{P}} \frac{1}{p^s} = +\infty,$$

hence the result.

1.3. **Summation by Parts.** Let $a : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function, let $0 < y < x$ and let $f : [y, x] \rightarrow \mathbb{C}$ be a function with continuous derivative on $[y, x]$. Then

$$\sum_{y < n \leq x} a_n f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt,$$

where $A(x) = \sum_{n \leq x} a_n$.

1.4. **Exercise.** Prove that for every $\delta > 0$,

$$\pi(x) := |\{p \in \mathcal{P} \mid p \leq x\}|$$

is bigger than $\frac{x}{(\log x)^{1+\delta}}$ for some sufficiently large x .

1.5. **Solution.** Let a_n be the prime indicator function, i.e.

$$a_n := \begin{cases} 1, & \text{if } n \text{ is prime} \\ 0, & \text{if } n \text{ is not a prime.} \end{cases} .$$

Using summation by parts we note that,

$$\sum_{p \leq x} \frac{1}{p} = \sum_{3/2 < n \leq x} \frac{a_n}{n} = \frac{\pi(x)}{x} + \int_{3/2}^x \frac{\pi(t)}{t^2} dt.$$

If the claim is false i.e. for all sufficiently large x , $\pi(x) \leq x/(\log x)^{1+\delta}$ then from the above,

$$\sum_{p \leq x} \frac{1}{p} \leq \frac{1}{(\log x)^{1+\delta}} + C + \frac{1}{(\log x)^\delta},$$

for some constant C . The RHS of the above tends to C as $x \rightarrow \infty$ contradicting Exercise 1.1, hence the result.

1.6. **Exercise.** Prove that for $\Re(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx,$$

where $\{x\}$ is the fractional part of x . Using this show that $\zeta(s)$ has meromorphic continuation to $\Re(s) > 0$ with a simple pole at $s = 1$.

1.7. **Solution.** Let $\Re(s) > 1$. Then using the summation by parts as following.

$$\begin{aligned} \sum_{n \leq x} \frac{1}{n^s} &= \frac{[x]}{x^s} + s \int_1^x \frac{[t]}{t^{s+1}} dt = \frac{1}{x^{s-1}} - \frac{\{x\}}{x^s} + s \int_1^x t^{-s} dt - s \int_1^x \frac{\{t\}}{t^{s+1}} dt \\ &= \frac{s}{s-1} - s \int_1^x \frac{\{t\}}{t^{s+1}} dt + O(x^{-\Re(s)} + x^{-\Re(s)+1}). \end{aligned}$$

Letting $x \rightarrow \infty$, as $\Re(s) > 1$, we conclude that

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{\{x\}}{x^{s+1}} dx.$$

We now note that the integral right hand side is well defined for $\Re(s) > 0$ and is holomorphic in s . As $\frac{s}{s-1}$ is a meromorphic function with simple pole at $s = 1$ and residue 1, we conclude the meromorphic continuation of $\zeta(s)$ to $\Re(s) > 0$.

1.8. **Exercise.** Prove that the Gamma function, which is defined for $\Re(s) > 0$ by

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1} dt$$

has analytic continuation to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple pole at each non-positive integer. Find the residues of the Gamma function at those poles.

Hint: First prove that $\Gamma(s+1) = s\Gamma(s)$.

1.9. **Solution.** By integration by parts we see that

$$\Gamma(s + 1) = \int_0^\infty e^{-t} t^{s+1} \frac{dt}{t} = \int_0^\infty e^{-t} s t^s \frac{dt}{t} = s\Gamma(s),$$

for $\Re(s) > 0$. Thus $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ extends definition of $\Gamma(s)$ to $\Re(s) > -1$ meromorphically with pole at $s = 0$ as

$$\lim_{s \rightarrow 0^+} \int_0^\infty e^{-t} t^s \frac{dt}{t} = +\infty.$$

The pole is simple, as $\lim_{s \rightarrow 0} s\Gamma(s) = 1$, and with residue 1. Similarly $\Gamma(s)$ can be extended to all $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ with simple poles at $s = -n$, $n \in \mathbb{N}$ with residue,

$$\lim_{s \rightarrow -n} (s + n)\Gamma(s) = \lim_{s \rightarrow -n} \frac{\Gamma(s + n + 1)}{(s + n - 1) \dots s} = \frac{(-1)^n}{n!}.$$

1.10. **Exercise.** Prove the Poisson summation formula: Let $f \in \mathcal{S}(\mathbb{R})$ be in the Schwartz class. Prove that

$$\sum_{n \in \mathbb{Z}} f(n + u) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e(nu).$$

Note: Putting $u = 0$ we get the usual Poisson summation formula.

1.11. **Solution.** Let

$$F(x) := \sum_{n \in \mathbb{Z}} f(n + x)$$

which is a function on $L^1(\mathbb{R}/\mathbb{Z})$ so has a Fourier expansion of the form

$$F(x) = \sum_{n \in \mathbb{Z}} e(nx) \hat{F}(n).$$

Here

$$\begin{aligned} \hat{F}(n) &= \int_0^1 F(x) e(-nx) dx = \sum_{n \in \mathbb{Z}} \int_0^1 \sum_{m \in \mathbb{Z}} f(m + x) e(-nx) dx \\ &= \sum_{m \in \mathbb{Z}} \int_m^{m+1} f(x) e(-nx) dx = \int_{-\infty}^\infty f(x) e(-nx) dx = \hat{f}(n), \end{aligned}$$

this provides the result.

1.12. **Exercise.** Recall that,

$$G(1, N) := \sum_{n \pmod N} e(n^2/N).$$

Prove that

- (1) For any odd positive integer N , $G(1, N^2) = N$ and $G(1, N^3) = NG(1, N)$.
- (2) For every positive integer N , $G(1, N) = \frac{1+i^{-N}}{1-i} \sqrt{N}$.

1.13. **Solution.** (1) is elementary. We can parametrize the residue class of N^k by

$$\{a_1 N^{k-1} + a_2 N^{k-2} + \cdots + a_k \mid 0 \leq a_i \leq N-1\}.$$

Using this we have,

$$\begin{aligned} G(1, N^2) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(aN+b)^2}{N^2}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \sum_{a=0}^{N-1} e\left(\frac{2ab}{N}\right) \\ &= \sum_{b=0}^{N-1} e(b^2/N^2) \delta_{b=0} N = N. \end{aligned}$$

Similarly,

$$\begin{aligned} G(1, N^3) &= \sum_{a=0}^{N-1} \sum_{b=0}^{N-1} \sum_{c=0}^{N-1} e\left(\frac{(aN^2 + bN + c)^2}{N^3}\right) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) \sum_{a=0}^{N-1} e(2ac/N) \\ &= \sum_{c=0}^{N-1} \sum_{b=0}^{N-1} e\left(\frac{(bN+c)^2}{N^3}\right) N \delta_{c=0} = NG(1, N). \end{aligned}$$

For the second part we use the Poisson summation formula. First we note the function

$$f(x) := 1_{[0, N]} e(x^2/N)$$

is a function which is continuous on $(0, N)$ and has continuity only from one side at $x = 0, N$. From the Fourier theory we know that the Fourier series of f at $x = 0$ would converge to $\frac{f(0+) + f(0-)}{2} = f(0+)/2$. and similarly, at $x = N$ to $f(N-)/2$. Thus using the (modified) Poisson summation formula and using that $f(0+) = f(N-)$ we get that,

$$\begin{aligned} \sum_{n=0}^N e(N^2/N) &= \frac{f(0+)}{2} + \sum_{n=1}^{N-1} f(n) + \frac{f(N-)}{2} \\ &= \sum_{n \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(nx) dx = \sum_{n \in \mathbb{Z}} \int_0^N e(x^2/N + nx) dx. \end{aligned}$$

Thus,

$$G(1, N) = N \sum_{n \in \mathbb{Z}} \int_0^1 e(Nx^2 + nNx) dx = N \sum_{n \in \mathbb{Z}} e(-Nn^2/4) \int_0^1 e(N(x + n/2)^2)$$

Noting that

$$e(-Nn^2/4) = \begin{cases} 1, & \text{if } n \text{ is even,} \\ i^{-N}, & \text{if } n \text{ is odd.} \end{cases}$$

and dividing the above sum into odd and even parts we get that,

$$\begin{aligned} G(1, N) &= N \sum_{n \in \mathbb{Z}} \int_n^{1+n} e(Nx^2) dx + Ni^{-N} \sum_{n \in \mathbb{Z}} \int_{n-1/2}^{n+1/2} e(nx^2) dx \\ &= \sqrt{N}(1 + i^{-N}) \int_{-\infty}^{\infty} e(y^2) dy. \end{aligned}$$

The last integral can be checked convergent and we call it C . Thus,

$$G(1, N) = \sqrt{N}C(1 + i^{-N}).$$

Checking that, $G(1, 1) = 1$, we conclude the result.

1.14. Dirichlet Character. A *Dirichlet character with modulus q* is a character

$$\chi : \mathbb{Z}/q\mathbb{Z}^\times \rightarrow \mathbb{C}^\times$$

extended to \mathbb{Z} by making it q -periodic and defining $\chi(a) = 0$ for $(a, q) > 1$. Associated to each character χ , in addition to its modulus q , is a natural number q' , its conductor. The *conductor q'* is the smallest divisor of q such that χ can be written as $\chi = \chi' \chi_0$, where χ_0 is the trivial Dirichlet character mod q and χ' is a character of modulus q' . If a character has conductor equal to its modulus then it is called a *primitive Dirichlet character*. Check that, for a primitive Dirichlet character χ mod q one has

$$\frac{1}{q} \sum_{a \pmod q} \chi(ma + b) = \begin{cases} \chi(b), & \text{if } q \mid m \\ 0, & \text{if } q \nmid m. \end{cases}$$

The above is not true for a non-primitive character.

1.15. Exercise. Let χ be a primitive Dirichlet character mod q and $f \in L^1(\mathbb{R})$. Prove that

$$\sum_{m \in \mathbb{Z}} f(m) \chi(m) = \frac{G(\chi)}{q} \sum_{n \in \mathbb{Z}} \hat{f}(n/q) \bar{\chi}(n),$$

where $G(\chi)$ is the Gauss sum attached to χ defined by

$$G(\chi) := \sum_{a \pmod q} \chi(a) e(a/q).$$

Hint: Use the Poisson summation formula.

1.16. **Solution.** First we prove the following. Let $v \in \mathbb{R}$ and $u \in \mathbb{R}^+$. Then using the Poisson summation formula,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} f(um + v) &= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(ux + v) e(-mx) dx \\ &= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} f(x) e(-m(x - v)/u) \frac{dx}{u} \\ &= \frac{1}{u} \sum_{m \in \mathbb{Z}} \hat{f}(m) e(mv/u). \end{aligned}$$

Using the above we get that,

$$\begin{aligned} \sum_{m \in \mathbb{Z}} f(m) \chi(m) &= \sum_{m \in \mathbb{Z}} \sum_{\substack{a \\ a \pmod{q}}} \chi(a) f(mq + a) \\ &= \sum_{\substack{a \\ a \pmod{q}}} \chi(a) \frac{1}{q} \sum_{m \in \mathbb{Z}} \hat{f}(m) e(ma/q) \\ &= \frac{G(\chi)}{q} \sum_{m \in \mathbb{Z}} \hat{f}(m) \bar{\chi}(m). \end{aligned}$$

Here in the last line we have used that for a primitive Dirichlet character χ ,

$$\sum_{\substack{a \\ a \pmod{q}}} \chi(a) e(am/q) = \bar{\chi}(m) G(\chi).$$

This can be seen as follows. Let $(m, q) = 1$. Then,

$$\bar{\chi}(m) G(\chi) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(am^{-1}) e(a/q) = \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} \chi(a) e(am/q).$$

If $(m, q) > 1$ then it follows from the fact that $\chi(m) = 0$ and

$$\sum_{\substack{a \\ a \pmod{q}}} \chi(a) e(am/q) = \sum_{\substack{y \\ y \pmod{q/(q,m)}}} e(y m/q) \sum_{\substack{x \\ x \pmod{q}}} \chi(xq/d + y) = 0.$$