D-MATH | Commutative Algebra | HS 2017
Prof. Paul Nelson

# Exercise Sheet 1

## Radical ideals, Local Rings and Affine Varieties

Let $A$ be a ring, $k$ an algebraically closed field and $n > 0$ an integer.

1. Let $\mathfrak{a} \subset A$ be an ideal. Show that its radical $r(\mathfrak{a})$ is an ideal. Furthermore, prove:
   (a) $r(\mathfrak{a}) \supset \mathfrak{a}$
   (b) $r(r(\mathfrak{a})) = r(\mathfrak{a})$
   (c) $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$
   (d) $r(\mathfrak{a}) = (1) \iff \mathfrak{a} = (1)$
   (e) $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$
   (f) if $\mathfrak{p} \subset A$ is a prime ideal, then $r(\mathfrak{p}^k) = \mathfrak{p}$ for all $k > 0$

2. Consider the polynomial ring $A[X]$. Let $f = \sum_{i=0}^{n} a_i X^i \in A[X]$ be a polynomial. Prove:
   (a) $f$ is a unit in $A[X]$ if and only if $a_0$ is a unit in $A$ and $a_1, \ldots, a_n$ are nilpotent.
   (b) $f$ is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent.
   (c) $f$ is a zero-divisor if and only if there exists $a \neq 0$ in $A$ such that $af = 0$.

3. Fix an element $x_0 \in \mathbb{R}^n$. Denote by $\mathfrak{U} := \{U \subset \mathbb{R}^n \text{ open } \mid x_0 \in U\}$ the set of open neighbourhoods of $x_0$ and define the set
$$S := \{(U, f) \mid U \in \mathfrak{U},\ f : U \to \mathbb{R} \text{ continuous}\}.$$
   We define an equivalence relation on $S$ as follows: two elements $(U, f), (V, g) \in S$ are equivalent if and only if there is an open neighbourhood $W \subset U \cap V$ of $x_0$ such that $f|_W = g|_W$. We denote the set of equivalence classes of $S$ by $R$. It is called *ring of germs* of continuous functions. Prove that $R$ is a local ring.

4. Show that the Zariski topology on $\mathbb{C}^n$ is coarser than the usual topology.

5. Let $X \subset k^n$ be a subset. Show that $I(X)$ is an ideal in $k[X_1, \ldots, X_n]$ and it is radical.

6. Let $X, X' \subset k^n$ and $S, S' \subset k[X_1, \ldots, X_n]$ be subsets. Show:

(a) $X \subset V(S) \iff S \subset I(X)$

(b) $V(S \cup S') = V(S) \cap V(S')$

(c) $I(X \cup X') = I(X) \cap I(X')$

(d) $S \subset S' \Rightarrow V(S) \supset V(S')$

(e) $X \subset X' \Rightarrow I(X) \supset I(X')$

(f) $S \subset I(V(S))$

(g) $X \subset V(I(X))$

(h) $V(S) = V(I(V(S)))$

(i) $I(X) = I(V(I(X)))$