D-MATH
Prof. Paul Nelson

Commutative Algebra

HS 2017

Exercise Sheet 11

Integral Ring Extensions

1. Let $A \hookrightarrow B$ be an integral ring extension. Let $f : A \to k$ be a homomorphism to an algebraically closed field k. Prove that f can be extended to a homomorphism $B \to k$, which restricts to f on A.

2. Let $A \hookrightarrow B$ be an integral ring extension. Prove:

   (a) If $x \in A$ is a unit in B, then it is a unit in A.

   (b) The Jacobson radical of A is the contraction of the Jacobson radical of B.

3. Let $A \hookrightarrow B$ be an integral ring extension. Let $\mathfrak{n} \subset B$ be a maximal ideal and denote $\mathfrak{m} := \mathfrak{n} \cap A$ for the corresponding maximal ideal in A. Is $B_{\mathfrak{n}}$ necessarily integral over $A_{\mathfrak{m}}$?

4. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{C}$ is not Noetherian.

5. Let A be an integral domain with field of fractions K. Let L/K be an algebraic field extension and B be the integral closure of A in L. Prove that the field of fractions of B is equal to L.