D-MATH
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Commutative Algebra

HS 2017

Exercise Sheet 12

Integral Extensions

1. Let A be a normal integral domain with field of fractions K. Let L/K be an algebraic field extension. Prove that an element $b \in L$ is integral over A if and only if its minimal polynomial over K lies in $A[X]$.

2. Let A be a ring and G a finite group of automorphisms of A. Let A^G denote the subring of G-invariants of A, i.e. all $x \in A$ such that $\sigma(x) = x$ for all $\sigma \in G$.

   (a) Prove that A is integral over A^G.

   (b) If moreover A is a normal integral domain with field of fractions K and let L/K be a Galois extension with Galois group G. Let B denote the integral closure of A in L. Prove that $\sigma(B) = B$ for all $\sigma \in G$ and $B^G = A$.

3. Let K be a field of characteristic zero. Find an explicit solution of Noether's Normalization Lemma for the following K-algebras:

   (a) $K[X, Y]/(XY)$

   (b) $K[X, Y, Z, W]/(XY - ZW)$

   (c) $K[X, Y, Z]/((XY - 1) \cap (Y, Z))$

4. Prove that every unique factorisation domain is normal.

5. Let R be a ring and $A \subset B$ be R-algebras. Suppose that $B_\mathfrak{p}$ is integral over $A_\mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset R$. Prove that B is integral over A.