D-MATH
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Commutative Algebra

HS 2017

# Exercise Sheet 13

## Valuation Rings

1. Let $A$ be an integral domain. Prove:
   (a) $A$ is a valuation ring if and only if for all pairs of ideals $\mathfrak{a}, \mathfrak{b} \subset A$ we have $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$.
   (b) If $A$ is a valuation ring and $\mathfrak{p} \subset A$ a prime ideal, then $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are both valuation rings.
2. Let $A$ be a valuation ring with field of fractions $K$. Prove that every ring $B$ with $A \subset B \subset K$ is a localisation of $A$ at a prime ideal.
3. Let $K$ be a field and consider the field $K(X)$.
   (a) Let $P \in K[X]$ be irreducible. Construct a normalized discrete valuation $\nu_P : K(X)^* \to \mathbb{Z}$ such that its valuation ring is $K[X]_{(P)}$.
   (b) Prove that $\tau : K(X)^* \to \mathbb{Z}$ defined by $\tau(\frac{f}{g}) = \deg(g) - \deg(f)$ is another valuation.
   (c) Prove that the valuations $\tau$ and $\nu_P$ for all irreducible polynomials $P \in K[X]$ are precisely all non-trivial valuations on $K(X)$ which are trivial on $K$.
4. Prove that an algebraically closed field does not admit any non-trivial discrete valuations.
5. Let $a \in \mathbb{C}$. Let $A$ be the ring of functions, which are holomorphic in some disc centered at $a$. Prove that $A$ is a discrete valuation ring and find a uniformizer.
6. Describe the spectrum of a discrete valuation ring.