D-MATH
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Commutative Algebra

HS 2017

Exercise Sheet 2

Extensions and Contractions, Modules, Spectrum of a Ring

1. Consider rings A, B and a ring homomorphism $\varphi : A \to B$. As in the lecture, denote:

$$C := \{\varphi^*(\mathfrak{b}) \mid \mathfrak{b} \subset B\} \subset A$$
$$E := \{\varphi_*(\mathfrak{a}) \mid \mathfrak{a} \subset A\} \subset B$$

for the set of contracted ideals and extended ideals, respectively. Show that C is closed under intersections, taking radicals and ideal quotients of ideals and E is closed under sums and products of ideals. More precisely, show that:

   (a) for all $\mathfrak{a}, \mathfrak{b} \in C$ we have $\mathfrak{a} \cap \mathfrak{b} \in C$, $r(\mathfrak{a}) \in C$ and $(\mathfrak{a} : \mathfrak{b}) \in C$.

   (b) for all $\mathfrak{a}, \mathfrak{b} \in E$ we have $\mathfrak{a} + \mathfrak{b} \in E$ and $\mathfrak{a}\mathfrak{b} \in E$.

2. Let A be a ring and $\mathfrak{a} \subset A$ be an ideal that is contained in the Jacobson radical of A. Let M, N be A-modules, where N is finitely generated, and let $\varphi : M \to N$ be an A-module homomorphism. Consider the induced homomorphism

$$\varphi_{\mathfrak{a}} : M/\mathfrak{a}M \to N/\mathfrak{a}N$$

Prove that if $\varphi_{\mathfrak{a}}$ is surjective, then φ is surjective.

3. Let k be a field and $0 \to M_0 \to \cdots \to M_n \to 0$ be an exact sequence of finite dimensional k-vector spaces and k-linear maps. Prove that

$$\sum_{i=0}^{n} (-1)^i \dim_k(M_i) = 0$$

4. Prove the 4-Lemma by diagram chasing: If the rows of the commutative diagram of A-modules and A-module homomorphisms

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\ M_1' \arrow[r] & M_2' \arrow[r] & M_3' \arrow[r] & M_4' \end{tikzcd}$$

are exact, then the following holds:

   (a) If α is surjective, and β and δ are injective, then γ is injective;

   (b) if δ is injective, and α and γ are surjective, then β is surjective.

5. Prove the 3×3-lemma: If

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_1' \arrow[r] \arrow[d] & M_2' \arrow[r] \arrow[d] & M_3' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1'' \arrow[r] \arrow[d] & M_2'' \arrow[r] \arrow[d] & M_3'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a commutative diagram of A-modules and A-module homomorphisms, and all columns and the middle row are exact, then the top row is exact if and only if the bottom row is exact.

6. In this exercise, we generalize the notion of an affine variety introduced in the lecture. Let A be a ring. We denote by $\operatorname{spec}(A)$ the set of all prime ideals of A. For a subset $S \subset A$ define

$$V(S) := \{\mathfrak{p} \in \operatorname{spec}(A) \mid S \subset \mathfrak{p}\}$$

Show that:

(a) If $\mathfrak{a} \subset A$ is the ideal generated by S, then $V(S) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

(b) $V(0) = \operatorname{spec}(A)$ and $V(1) = \varnothing$.

(c) For a family of subsets $(S_i)_{i \in I} \subset A$ we have $V(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} V(S_i)$.

(d) For finitely many ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n \subset A$ we have $V(\bigcap_{i=1}^n \mathfrak{a}_i) = \bigcup_{i=1}^n V(\mathfrak{a}_i)$.

This shows that the subsets $(V(S))_{S \subset A}$ form the closed sets of a topology on $\operatorname{spec}(A)$, called *Zariski* topology. We call the topological space $\operatorname{spec}(A)$ the *(prime) spectrum* of A.