D-MATH
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Commutative Algebra

HS 2017

Exercise Sheet 3

Tensor Product, Modules, Spectrum of a Ring

1. Let A be a local ring and M, N two finitely generated A-modules. Prove that $M \otimes_A N = 0$ implies $M = 0$ or $N = 0$. Give an example of modules over a non-local ring which do not have this property.

2. Let A be a ring. Prove the following:

   (a) If M and N are flat A-modules, then so is $M \otimes_A N$.

   (b) If B is a flat A-algebra and M a flat B-module, then M is flat as an A-module.

3. Let A be a ring. Consider a short exact sequence of A-modules and homomorphisms $0 \to M' \to M \to M'' \to 0$. Prove that if M' and M'' are finitely generated, then so is M.

4. Let A be a ring. Prove that for any three A-modules M_1, M_2, M and homomorphisms $M_1 \xrightarrow{f} M \xleftarrow{g} M_2$ there exists an A-module P and homomorphisms $M_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_2} M_2$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & M_2 \arrow[d, "g"] \\ M_1 \arrow[r, "f"] & M \end{tikzcd}$$

commutes and with the following universal property: for any A-module N and homomorphisms $M_1 \xleftarrow{u} N \xrightarrow{v} M_2$ such that $f \circ u = g \circ v$ there exists a unique homomorphism $h : N \to P$ making the whole diagram commute:

$$\begin{tikzcd} N \arrow[dr, dotted, "h"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\ & P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & M_2 \arrow[d, "g"'] \\ & M_1 \arrow[r, "f"] & M \end{tikzcd}$$

Finally, show that P is unique up to a unique isomorphism.

[Hint: Look at a submodule of $M_1 \oplus M_2$.]

5. Let A be a ring. Recall the definition of the prime spectrum of a ring from exercise sheet 2. For every element $f \in A$ denote $D(f)$ for the open complement of $V((f))$ in $\operatorname{spec}(A)$. Show that these sets form a basis of open sets for the Zariski topology on $\operatorname{spec}(A)$. Furthermore, prove:

   (a) $\forall f, g \in A$ we have $D(f) \cap D(g) = D(fg)$

   (b) $D(f) = \varnothing$ if and only if f is nilpotent

   (c) $D(f) = \operatorname{spec}(A)$ if and only if f is a unit

   (d) $\operatorname{spec}(A)$ is quasicompact

   These open sets are called *basic open sets* of $\operatorname{spec}(A)$.