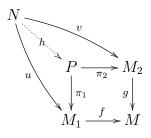
Exercise Sheet 3

TENSOR PRODUCT, MODULES, SPECTRUM OF A RING

- 1. Let A be a local ring and M, N two finitely generated A-modules. Prove that $M \otimes_A N = 0$ implies M = 0 or N = 0. Give an example of modules over a non-local ring which do not have this property.
- 2. Let A be a ring. Prove the following:
 - (a) If M and N are flat A-modules, then so is $M \otimes_A N$.
 - (b) If B is a flat A-algebra and M a flat B-module, then M is flat as an A-module.
- 3. Let A be a ring. Consider a short exact sequence of A-modules and homomorphisms $0 \to M' \to M \to M'' \to 0$. Prove that if M' and M'' are finitely generated, then so is M.
- 4. Let A be a ring. Prove that for any three A-modules M_1, M_2, M and homomorphisms $M_1 \xrightarrow{f} M \xleftarrow{g} M_2$ there exists an A-module P and homomorphisms $M_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_2} M_2$ such that the diagram

$$\begin{array}{c} P \xrightarrow{\pi_2} M_2 \\ \downarrow_{\pi_1} & \downarrow_g \\ M_1 \xrightarrow{f} M \end{array}$$

commutes and with the following universal property: for any A-module N and homomorphisms $M_1 \stackrel{u}{\leftarrow} N \stackrel{v}{\rightarrow} M_2$ such that $f \circ u = g \circ v$ there exists a unique homomorphism $h: N \to P$ making the whole diagram commute:



Finally, show that P is unique up to a unique isomorphism. [Hint: Look at a submodule of $M_1 \oplus M_2$.]

- 5. Let A be a ring. Recall the definition of the prime spectrum of a ring from exercise sheet 2. For every element $f \in A$ denote D(f) for the open complement of V((f)) in spec(A). Show that these sets form a basis of open sets for the Zariski topology on spec(A). Furthermore, prove:
 - (a) $\forall f, g \in A$ we have $D(f) \cap D(g) = D(fg)$
 - (b) $D(f) = \emptyset$ if and only if f is nilpotent
 - (c) $D(f) = \operatorname{spec}(A)$ if and only if f is a unit
 - (d) $\operatorname{spec}(A)$ is quasicompact

These open sets are called *basic open sets* of spec(A).