D-MATH Commutative Algebra HS 2017
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Exercise Sheet 4

Localisation, Splitting Lemma, Irreducible Variety

1. Let A be a ring ~~reduced ring (i.e. without any nonzero nilpotent elements)~~. Let M be a finitely generated A-module and let $f : M \to M$ be a surjective module homomorphism. Then f is also injective.

   Remark: The intended proof for A reduced did not work, but there is a more general proof. Many apologies for this inconvenience!

2. Let A be a ring such that every localisation $A_{\mathfrak{p}}$ of A with respect to a prime ideal $\mathfrak{p} \subset A$ has no nonzero nilpotent elements. Prove that A has no nonzero nilpotent elements. Is the same true for zero-divisors?

3. Let A be a ring. Let T, S be two multiplicatively closed subsets and let U be the image of T in $S^{-1}A$. Prove that $(ST)^{-1}A$ is isomorphic to $U^{-1}S^{-1}A$.

4. Let A be an integral domain and M an A-module. Prove that the following are equivalent:

   (a) M is torsion-free.

   (b) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals $\mathfrak{p} \subset A$.

   (c) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals $\mathfrak{m} \subset A$.

5. (Splitting Lemma) Let A be a ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of A-modules. The sequence is called *split* if there is an isomorphism $M \to M' \oplus M''$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \longrightarrow & 0 \\ & & \| & & \downarrow \cong & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & M' \oplus M'' & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

   commutes, where the homomorphisms in the lower row are the inclusion and projection respectively.

   Prove the splitting lemma, i.e. that the following are equivalent:

   (a) The short exact sequence splits.

   (b) There is a homomorphism $i : M'' \to M$ such that $v \circ i = \mathrm{id}_{M''}$.

   (c) There is a homomorphism $s : M \to M'$ such that $s \circ u = \mathrm{id}_{M'}$.

6. A topological space is called *irreducible* if it is non-empty and every two non-empty open subsets have a non-empty intersection. Prove that for $\mathrm{spec}(A)$ the following are equivalent:

   (a) $\mathrm{spec}(A)$ is irreducible.

   (b) The nilradical of A is a prime ideal.

   (c) There is a dense point $x \in \mathrm{spec}(A)$, i.e. the closure of $\{x\}$ is $\overline{\{x\}} = \mathrm{spec}(A)$.

   Remark: We call such a point as in (c) a *generic point*.