Exercise Sheet 6

Primary Decomposition & k-Algebras

Definition: Let A be a ring and $\mathfrak{a} \subset A$ an ideal which admits a primary decomposition. Let P be the set of associated prime ideals of $\mathfrak{a}$. The minimal elements of P with respect to inclusion are called *isolated prime ideals* of $\mathfrak{a}$, the others are called *embedded prime ideals*.

1. The power of a maximal ideal is a primary ideal. Show that the converse is not true: find an example of a ring A and a primary ideal $\mathfrak{a} \subset A$ such that its radical $r(\mathfrak{a})$ is a maximal ideal, but $\mathfrak{a}$ is not a power of $r(\mathfrak{a})$.

2. Let k be a field. Consider the polynomial ring $A := k[X, Y, Z]$ and the ideal $\mathfrak{a} := (X^2, XY, YZ, XZ) \subset A$. Find a minimal primary decomposition of $\mathfrak{a}$ and the associated prime ideals. Which components are isolated and which are embedded?

3. Let k be a field and A a finitely generated k-algebra. Prove the following statement: An ideal $\mathfrak{a} \subset A$ is a maximal ideal if and only if $\mathfrak{a}$ is prime and the quotient $A/\mathfrak{a}$ is a finite dimensional k-vector space.

4. Let k be a field and A a finitely generated k-algebra. Let $\mathfrak{a} \subset A$ be a radical ideal. Using the previous exercise, prove that the associated prime ideals of $\mathfrak{a}$ are all maximal if and only if $A/\mathfrak{a}$ is a finite dimensional k-vector space.

5. For $k := \mathbb{C}$, explain the geometry behind exercise 2.