Exercise Sheet 7

Primary Decomposition, Artinian Rings and Modules

Definition: Let A be a ring and $\mathfrak{p} \subset A$ a prime ideal of A. Denote $\varphi : A \to A_{\mathfrak{p}}$ for the localisation map. For an integer $n > 0$ we define the *n-th symbolic power of* $\mathfrak{p}$ to be the ideal $\mathfrak{p}^{(n)} := \varphi^* \varphi_*(\mathfrak{p}^n)$.

1. Let A be a ring and $\mathfrak{p} \subset A$ a prime ideal. Let $n > 0$ be an integer. Consider the n-th symbolic power $\mathfrak{p}^{(n)}$. Prove:

   (a) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal.

   (b) If $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component.

   (c) $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if $\mathfrak{p}^n$ is a primary ideal.

2. Let k be a field and consider the ring $A := k[X, Y, Z]$. Compute the ideal of A given by

   $$\mathfrak{a} := (Y^2 + XY - XZ - YZ, (X + Y)^2 + 2X) \cap ((X + Y)^2, X, Y^3 - Y^2 Z)$$

   and find a minimal primary decomposition of $\mathfrak{a}$.

   (*Hint:* Substitutions might be helpful.)

3. Let A be a ring and $\mathfrak{a} \subset A$ an ideal which admits a primary decomposition. Let $\mathfrak{p}$ be a maximal element of the set $\{(\mathfrak{a} : x) \mid x \in A \smallsetminus \mathfrak{a}\}$. Prove that $\mathfrak{p}$ is an associated prime ideal of $\mathfrak{a}$.

4. Let A be a ring and M an Artinian A-module. Let $f : M \to M$ be an injective module homomorphism. Prove that f is an isomorphism.

5. Let A be a ring and M a Noetherian A-module. Let $\mathfrak{a} \subset A$ be the annihilator of M, i.e. $\mathfrak{a} = \{x \in A \mid xM = 0\}$. Prove that the ring $A/\mathfrak{a}$ is Noetherian.

   Is the same true if we replaced Noetherian with Artinian?