D-MATH
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Commutative Algebra

HS 2017

Exercise Sheet 9

Dimension and Height

1. Let A be a ring. Prove the following statements:

   (a) For every prime ideal $\mathfrak{p} \subset A$ we have $\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) \leqslant \dim(A)$.

   (b) For every ideal $\mathfrak{a} \subset A$ we have $\mathrm{ht}(\mathfrak{a}) + \mathrm{coht}(\mathfrak{a}) \leqslant \dim(A)$.
   (Recall that $\mathrm{ht}(\mathfrak{a}) = \inf\limits_{\mathfrak{p} \supset \mathfrak{a} \text{ prime}} \mathrm{ht}(\mathfrak{p})$)

2. Give an example of a

   (a) non-Noetherian local ring A with maximal ideal $\mathfrak{m}$ and $\dim A > \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

   (b) Noetherian non-local ring A with a maximal ideal $\mathfrak{m}$ such that $\dim A > \mathrm{ht}(\mathfrak{m})$.

3. Let A be a ring and consider the polynomial ring in one variable $A[X]$. Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subset A[X]$ be two prime ideals such that their contraction to A is equal $\mathfrak{p} := \mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A$. Prove that $\mathfrak{p}_1 = \mathfrak{p}A[X]$. Deduce that for any three subsequent prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3 \subset A[X]$ their contractions $\mathfrak{p}_1 \cap A, \mathfrak{p}_2 \cap A, \mathfrak{p}_3 \cap A$ cannot all be equal.

   [Hint: Take a ring of fractions and use that $\dim(K[X]) = 1$ for every field K]

4. Consider the ring $R := \mathbb{C}[X, Y, Z]/(XY, XZ)$. Compute the height of the two maximal ideals $\mathfrak{m}_1 := (X - 1, Y, Z) \subset R$ and $\mathfrak{m}_2 := (X, Y - 1, Z) \subset R$. Interpret your result geometrically on the variety $V(XY, XZ) \subset \mathbb{C}^3$.