Commutative Algebra

Solutions Sheet 11

Integral Ring Extensions

1. Let $A \hookrightarrow B$ be an integral ring extension. Let $f : A \to k$ be a homomorphism to an algebraically closed field k. Prove that f can be extended to a homomorphism $B \to k$, which restricts to f on A.

   Solution: Let $\mathfrak{p} := \ker(f) \subset A$. Since k is a field, the ideal $\mathfrak{p}$ is prime. Using this and the universal property of the field of fractions we conclude that we can factor f as
   $$A \to A/\mathfrak{p} \to K \to k$$
   where K is the field of fractions of $A/\mathfrak{p}$. On the other hand, by Lying over there is a prime ideal $\mathfrak{q} \subset B$ with $\mathfrak{q} \cap A = \mathfrak{p}$ and $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$. Let L be the field of fractions of $B/\mathfrak{q}$. The inclusion $A/\mathfrak{p} \to B/\mathfrak{q}$ extends to an inclusion of fields $K \to L$. Since $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$, we conclude that L/K is an algebraic field extension. By a classical statement of algebra, we can thus lift the map $K \to k$ to a map $L \to k$. Together with the map $B \to B/\mathfrak{q} \to L$ this gives a lift of f.

2. Let $A \hookrightarrow B$ be an integral ring extension. Prove:

   (a) If $x \in A$ is a unit in B, then it is a unit in A.

   (b) The Jacobson radical of A is the contraction of the Jacobson radical of B.

   Solution:

   (a) Let $x \in A$ be a unit in B. Let $y \in B$ such that $xy = 1$. Since B is integral over A we conclude that y satisfies a polynomial equation
   $$y^n + a_{n-1}y^{n-1} + \cdots + a_1 y + a_0 = 0$$
   for elements $a_0, \ldots, a_{n-1} \in A$. Multiplying it by x^{n-1} gives the equation
   $$y + a_{n-1} + \cdots + a_1 x^{n-2} + a_0 x^{n-1} = 0$$
   And so $y \in A$.

   (b) By Lying over the maximal ideals of A are precisely the contractions of the maximal ideals of B. Since the Jacobson radical is the intersection of all maximal ideals we conclude the statement.

3. Let $A \hookrightarrow B$ be an integral ring extension. Let $\mathfrak{n} \subset B$ be a maximal ideal and denote $\mathfrak{m} := \mathfrak{n} \cap A$ for the corresponding maximal ideal in A. Is $B_{\mathfrak{n}}$ necessarily integral over $A_{\mathfrak{m}}$?

   Solution: No. Consider the rings $A := k[X^2 - 1] \subset k[X] =: B$, where k is a field. The ring B is integral over A, since the element X is integral over A. Let $\mathfrak{n} := (X - 1)$. This gives $\mathfrak{m} = (X - 1) \cap A = (X^2 - 1)$. We show that the element $\frac{1}{X+1}$ is not integral over $A_{\mathfrak{m}}$. Assume otherwise. Then there are polynomials $f_0, \ldots, f_n \in A$ and $g_0, \ldots, g_n \in A \setminus \mathfrak{m}$ such that

$$\sum_{i=0}^{n} \frac{f_i}{g_i (X+1)^i} = \frac{1}{(X+1)^{n+1}}$$

   But the $g_0, \ldots, g_n$ do not have a root at $X = \pm 1$. Thus the left hand side of the equation

$$\sum_{i=0}^{n} \frac{f_i (X+1)^{n-i}}{g_i} = \frac{1}{X+1}$$

   does not have a pole at $X = -1$. A contradiction.

4. Show that the integral closure of $\mathbb{Z}$ in $\mathbb{C}$ is not Noetherian.

   Solution: Denote by A the integral closure of $\mathbb{Z}$ in $\mathbb{C}$. Let $p \in \mathbb{Z}$ be a prime number. For $n \geqslant 1$ let a_n be a root of $X^{2^n} - p$ such that $a_{n+1}^2 = a_n$. By construction, the a_n are in A. We prove that the ideals $(a_n)_{n \geqslant 1}$ form a strictly ascending chain of ideals in A. We only need to show that $a_{n+1} \notin (a_n)$. Assume otherwise. Then $a_{n+1} = b a_n$ for some element $b \in A$. But then $a_n = a_{n+1}^2 = b^2 a_n^2$. This proves that a_n is a unit in A. Since A is integral over $\mathbb{Z}[a_n]$, we use exercise 2.(a) to conclude that a_n is a unit in $\mathbb{Z}[a_n]$. However, in the ring $\mathbb{Z}[a_n] \cong \mathbb{Z}[X]/(X^{2^n} - p)$ the element X is not invertible, because p is not invertible in $\mathbb{Z}$. A contradiction.

5. Let A be an integral domain with field of fractions K. Let L/K be an algebraic field extension and B be the integral closure of A in L. Prove that the field of fractions of B is equal to L.

   Solution: Since L is a field containing B, it also contains the field of fractions of B. Conversely, let $S := A \setminus (0)$. Since B is the integral closure of A in L we know that $S^{-1}B$ is the integral closure of $S^{-1}A = K$ in $S^{-1}L = L$. But the integral closure of K in L is L. Therefore we conclude that $L = S^{-1}B \subset \mathrm{frac}(B)$.