Solutions Sheet 12

Integral Extensions

1. Let A be a normal integral domain with field of fractions K. Let L/K be an algebraic field extension. Prove that an element $b \in L$ is integral over A if and only if its minimal polynomial over K lies in $A[X]$.

   Solution: Let $b \in L$ such that its minimal polynomial over K lies in $A[X]$. Since the minimal polynomial is monic we conclude that b is integral over A.

   Conversely assume that $b \in L$ is integral over A. Then there is a monic polynomial $P \in A[X]$ such that $P(b) = 0$. Let $M \in K[X]$ be the minimal polynomial of b over K. Since P is also an element in $K[X]$ we conclude that M divides P in $K[X]$. But all zeroes of P in an algebraic closure $\bar{L}$ of L are integral over A and so all zeroes of M in $\bar{L}$ are integral over A. Since the coefficients of a polynomial are polynomial expressions in the zeroes, we conclude that the coefficients of M are integral over A. Since they are in K and A is normal, we conclude that $M \in A[X]$.

2. Let A be a ring and G a finite group of automorphisms of A. Let A^G denote the subring of G-invariants of A, i.e. all $x \in A$ such that $\sigma(x) = x$ for all $\sigma \in G$.

   (a) Prove that A is integral over A^G.

   (b) If moreover A is a normal integral domain with field of fractions K and let L/K be a Galois extension with Galois group G. Let B denote the integral closure of A in L. Prove that $\sigma(B) = B$ for all $\sigma \in G$ and $B^G = A$.

   Solution:

   (a) Let $a \in A$. We define $g := \prod_{\sigma \in G}(X - \sigma(a))$. This is a polynomial, because G is a finite group. Furthermore $\sigma(g) = g$ for all $\sigma \in G$ and thus the coefficients of g are in A^G. So g is a monic polynomial in $A^G[X]$ with a as a zero, which proves the claim.

   (b) In the lecture, we have already seen that $\sigma(B) = B$ for all $\sigma \in G$. The inclusion $B^G \supset A$ follows from the fact that $B \supset A$ and G acts trivially on A. Conversely let $b \in B^G \subset B$. Then $b \in L^G = K$ and b is integral over A. Because A is normal, we conclude that $b \in A$. Hence $B^G \subset A$.

3. Let K be a field of characteristic zero. Find an explicit solution of Noether's Normalization Lemma for the following K-algebras:

   (a) $K[X, Y]/(XY)$

(b) $K[X,Y,Z,W]/(XY-ZW)$

(c) $K[X,Y,Z]/((XY-1)\cap(Y,Z))$

Solution: We find the solutions by following the proof of Noether's Normalisation Lemma.

(a) A solution is $K[X+Y]$.

(b) A solution is $K[X+Y,Z,W]$.

(c) A solution is $K[X+Y,X-Z]$.

4. Prove that every unique factorisation domain is normal.

   Solution: Let A be a unique factorisation domain with field of fractions K. Let $\frac{a}{b}\in K$ be integral over A with $a,b\in A$ relatively coprime. Thus there exists a monic polynomial $f=X^n+\sum_{i=0}^{n-1}c_iX^i\in A[X]$ such that $f(\frac{a}{b})=0$. Then we have $a^n+b\sum_{i=0}^{n-1}c_ia^ib^{n-i-1}=0$. This implies that b divides a^n. A contradiction to the unique factorisation property.

5. Let R be a ring and $A\subset B$ be R-algebras. Suppose that $B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}\subset R$. Prove that B is integral over A.

   Solution: Let $b\in B$. Let $\mathfrak{p}\subset R$ be a prime ideal. Since b is integral over $A_{\mathfrak{p}}$, we find elements $a_0,\dots,a_{n-1}\in A$ and $s_0,\dots,s_{n-1}\in R\smallsetminus\mathfrak{p}$ such that

$$b^n+(a_{n-1}/s_{n-1})b^{n-1}+\cdots+(a_0/s_0)=0$$

   Set $s_{\mathfrak{p}}:=s_0\dots s_{n-1}$. We multiply with $(s_{\mathfrak{p}})^n$ to find that $s_{\mathfrak{p}}b$ is integral over A. Let I be the ideal in R generated by $s_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R. We prove that $I=(1)$. If not, then I is contained in a maximal ideal $\mathfrak{m}$ of R. But $s_{\mathfrak{m}}\notin\mathfrak{m}$, which gives a contradiction. Since $I=(1)$ there is a finite linear combination $\sum_{\mathfrak{p}}' r_{\mathfrak{p}}s_{\mathfrak{p}}=1$ with $r_{\mathfrak{p}}\in R$. Multiplied by b this leads to $b=\sum_{\mathfrak{p}}' r_{\mathfrak{p}}s_{\mathfrak{p}}b$. As a sum of integral elements over A, we conclude that b is integral over A.