Solutions Sheet 13

Valuation Rings

1. Let A be an integral domain. Prove:

   (a) A is a valuation ring if and only if for all pairs of ideals $\mathfrak{a}, \mathfrak{b} \subset A$ we have $\mathfrak{a} \subset \mathfrak{b}$ or $\mathfrak{b} \subset \mathfrak{a}$.

   (b) If A is a valuation ring and $\mathfrak{p} \subset A$ a prime ideal, then $A_{\mathfrak{p}}$ and $A/\mathfrak{p}$ are both valuation rings.

   Solution:

   (a) Assume that A is a valuation ring and let $\mathfrak{a}, \mathfrak{b} \subset A$ be two ideals. If $\mathfrak{a} \not\subset \mathfrak{b}$, choose an element $f \in \mathfrak{a} \setminus \mathfrak{b}$. For all $g \in \mathfrak{b}$ we know that $\frac{f}{g} \notin A$, otherwise f would be in $\mathfrak{b}$. Since A is a valuation ring, we know that $\frac{g}{f} \in A$ and thus $g = \frac{g}{f} f \in \mathfrak{a}$. We conclude that $\mathfrak{b} \subset \mathfrak{a}$. Conversely, let $x := \frac{f}{g}$ be an element in the field of fractions of A. By assumption we have $(f) \subset (g)$ or $(g) \subset (f)$. Hence $x \in A$ or $x^{-1} \in A$.

   (b) By the inclusion preserving correspondence of ideals in A which contain $\mathfrak{p}$ (resp. are contained by $\mathfrak{p}$) and ideals in $A/\mathfrak{p}$ (resp. $A_{\mathfrak{p}}$), we conclude by (a) that $A/\mathfrak{p}$ (resp. $A_{\mathfrak{p}}$) is a valuation ring.

2. Let A be a valuation ring with field of fractions K. Prove that every ring B with $A \subset B \subset K$ is a localisation of A at a prime ideal.

   Solution: For every element $x \in K$ we have $x \in A$ or $x^{-1} \in A$, hence $x \in B$ or $x^{-1} \in B$, which proves that B is a valuation ring. Thus B has a unique maximal ideal $\mathfrak{n}$. Then $\mathfrak{p} := \mathfrak{n} \cap A$ is a prime ideal of A. Every element $a \in A \setminus \mathfrak{p}$ is invertible in B, hence $A_{\mathfrak{p}} \subset B$. Conversely, let $x \in B$. If $x \in A$, then $x \in A_{\mathfrak{p}}$. Otherwise $x^{-1} \in A \subset B$. But then $x^{-1} \in B^\times = B \setminus \mathfrak{n}$ and thus $x^{-1} \in A \setminus \mathfrak{p}$. Hence $x \in A_{\mathfrak{p}}$. We conclude that $A_{\mathfrak{p}} = B$.

3. Let K be a field and consider the field $K(X)$.

   (a) Let $P \in K[X]$ be irreducible. Construct a normalized discrete valuation $\nu_P : K(X)^* \to \mathbb{Z}$ such that its valuation ring is $K[X]_{(P)}$.

   (b) Prove that $\tau : K(X)^* \to \mathbb{Z}$ defined by $\tau(\frac{f}{g}) = \deg(g) - \deg(f)$ is another valuation.

   (c) Prove that the valuations τ and ν_P for all irreducible polynomials $P \in K[X]$ are precisely all non-trivial valuations on $K(X)$ which are trivial on K.

Solution:

(a) For an element $x \in K(X)^*$ let $f, g \in K[X] \smallsetminus \{0\}$ such that P does neither divide f nor g and such that $x = P^n \frac{f}{g}$ for some $n \in \mathbb{Z}$. Then we set $\nu_P(x) := n$. This is a well-defined map $K(X)^* \to \mathbb{Z}$. It follows directly that it is normalized. For $x, y \in K(X)^*$ let $f, g, h, k \in K[X] \smallsetminus \{0\}$ such that $x = P^{\nu_P(x)} \frac{f}{g}$ and $y = P^{\nu_P(y)} \frac{h}{k}$. Assume without loss of generality that $\nu_P(y) \geqslant \nu_P(x)$. Then we see that $\nu_P(xy) = \nu_P(x) + \nu_P(y)$ and

$$x + y = P^{\nu_P(x)} \left(\frac{fk + P^{\nu_P(y) - \nu_P(x)} hg}{gk} \right)$$

The denominator cannot be divisible by P, so we have $\nu_P(x + y) \geqslant \nu_P(x) = \min\{\nu_P(x), \nu_P(y)\}$. Thus ν_P is a valuation. By definition it follows directly that the valuation ring of ν_P is $K[X]_{(P)}$.

(b) Let $\varphi : K(X) \to K(X)$ be the field isomorphism induced by $\varphi(X) = \varphi(X^{-1})$. Let ν_X be the valuation from (a) for $P := X \in K[X]$. We observe that $\tau = \nu_X \circ \varphi$. Hence τ is a valuation.

(c) Let ν be a nontrivial valuation of $K(X)$ which is trivial on K. We want to find its valuation ring A. It certainly contains K. Suppose first that $X \in A$, so $K[X] \subset A$. Let $\mathfrak{m}$ be the maximal ideal of A. Then $\mathfrak{m} \cap K[X]$ is a prime ideal, which is non-zero, because otherwise $K(X) \subset A$, contrary to the nontriviality of ν. Since $K[X]$ is a principal ideal domain, there is an irreducible polynomial $P \in K[X]$ such that $(P) = \mathfrak{m} \cap K[X]$. We conclude that $K[X]_{(P)} \subset A$. Conversely, by exercise 2, we know that A is a localisation of $K[X]_{(P)}$ at a prime ideal, hence a localisation of $K[X]$ at a prime ideal contained in (P). But the only such prime ideals are (0) and (P), where the former is not possible by the assumption $A \neq K(X)$. Hence $A = K[X]_{(P)}$. Since also $\nu(P) = 1$, we conclude that the valuation ν is the same as ν_P defined in (a).

Suppose $X \notin A$. Then $X^{-1} \in A$, so $K[X^{-1}] \subset A$. We use the automorphism φ defined in the proof of (b) and see that $K[X] \subset \varphi(A)$. We do the same argument as before for $\varphi(A)$ with the addition that $P = X$ in this case, because $X^{-1} \notin \varphi(A)$. We then find that $\nu_X = \nu \circ \varphi$. Since φ is its own inverse we conclude that $\nu = \tau$.

4. Prove that an algebraically closed field does not admit any non-trivial discrete valuations.

Solution: Let ν be a valuation of an algebraically closed field k. Pick $a \in k$ such that $\nu(a) > 0$. Then $\sqrt{a} \in k$, since k is algebraically closed and $\nu(\sqrt{a}) = \frac{1}{2}\nu(a)$. Doing this repeatedly, there are elements in k with arbitrary small valuations. Hence the valuation cannot be discrete.

5. Let $a \in \mathbb{C}$. Let A be the ring of functions, which are holomorphic in some disc centered at a. Prove that A is a discrete valuation ring and find a uniformizer.

   Solution: By a shift we can assume without loss of generality that $a = 0$. Via taylor series we can identify A with the ring $\mathbb{C}[[X]]$. Its field of fractions is $\mathbb{C}((X))$. We know that a power series is invertible if and only if its constant term is non-zero. Thus every non-zero element $x \in \mathbb{C}((X))$ can be written as $x = X^n f$, where f is an invertible element of $\mathbb{C}[[X]]$, for some $n \in \mathbb{Z}$. We set $\nu(x) := n$. One can directly check that this defines a discrete valuation on $\mathbb{C}((X))$ with valuation ring $\mathbb{C}[[X]]$. Furthermore, the element X is a uniformizer.

6. Describe the spectrum of a discrete valuation ring.

   Solution: A discrete valuation ring A has only two prime ideals, namely the zero ideal (0) and the maximal ideal $\mathfrak{m}$. We have that $V(0) = \operatorname{spec}(A)$ and $V(\mathfrak{m}) = \mathfrak{m}$ are the only closed subsets of $\operatorname{spec}(A)$. Hence $\varnothing$ and (0) are the only open subsets. We conclude that $\operatorname{spec}(A)$ consists of two points, one of which is closed, but not open and the other one is open, but not closed.