D-MATH
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Commutative Algebra

HS 2017

# Solutions Sheet 2

EXTENSIONS AND CONTRACTIONS, MODULES, SPECTRUM OF A RING

1. Consider rings $A, B$ and a ring homomorphism $\varphi : A \to B$. As in the lecture, denote:

$$C := \{\varphi^*(\mathfrak{b}) \mid \mathfrak{b} \subset B\} \subset A$$
$$E := \{\varphi_*(\mathfrak{a}) \mid \mathfrak{a} \subset A\} \subset B$$

for the set of contracted ideals and extended ideals, respectively. Show that $C$ is closed under intersections, taking radicals and ideal quotients of ideals and $E$ is closed under sums and products of ideals. More precisely, show that:

(a) for all $\mathfrak{a}, \mathfrak{b} \in C$ we have $\mathfrak{a} \cap \mathfrak{b} \in C$, $r(\mathfrak{a}) \in C$ and $(\mathfrak{a} : \mathfrak{b}) \in C$.

(b) for all $\mathfrak{a}, \mathfrak{b} \in E$ we have $\mathfrak{a} + \mathfrak{b} \in E$ and $\mathfrak{a}\mathfrak{b} \in E$.

*Solution*:

(a) Let $\mathfrak{c}, \mathfrak{d} \subset B$ be two ideals. The identity $\varphi^*(\mathfrak{c} \cap \mathfrak{d}) = \varphi^*(\mathfrak{c}) \cap \varphi^*(\mathfrak{d})$ follows by set theory and implies that the intersection of two contracted ideals is again contracted.

Next, we show that $\varphi^*(r(\mathfrak{c})) = r(\varphi^*(\mathfrak{c}))$. An element $f \in A$ is in $\varphi^*(r(\mathfrak{c}))$ if and only if there is an integer $n > 0$ such that $\varphi(f)^n = \varphi(f^n) \in \mathfrak{c}$. This is the case if and only if $f^n \in \varphi^*(\mathfrak{c})$ for some $n > 0$ which is equivalent to $f \in r(\varphi^*(\mathfrak{c}))$. This proves that the radical of a contracted ideal is again a contracted ideal.

Finally, we show that $\varphi^*((\mathfrak{c} : \varphi_*\varphi^*(\mathfrak{d}))) = (\varphi^*(\mathfrak{c}) : \varphi^*(\mathfrak{d}))$. Let $f \in \varphi^*((\mathfrak{c} : \varphi_*\varphi^*(\mathfrak{d})))$. Then $\varphi(f)\varphi_*\varphi^*(\mathfrak{d}) \subset \mathfrak{c}$, so by the property $\varphi^*\varphi_*\varphi^*(\mathfrak{d}) = \varphi^*(\mathfrak{d})$ we conclude $f\varphi^*(\mathfrak{d}) \subset \varphi^*(\mathfrak{c})$ and thus $f \in (\varphi^*(\mathfrak{c}) : \varphi^*(\mathfrak{d}))$. Conversely let $f \in (\varphi^*(\mathfrak{c}) : \varphi^*(\mathfrak{d}))$. Then $f\varphi^*(\mathfrak{d}) \subset \varphi^*(\mathfrak{c})$, which implies that $\varphi(f)\varphi(\varphi^*(\mathfrak{d})) \subset \mathfrak{c}$. Since $\mathfrak{c}$ is an ideal we conclude $\varphi(f)\varphi_*(\varphi^*(\mathfrak{d})) \subset \mathfrak{c}$. Hence $f \in \varphi^*((\mathfrak{c} : \varphi_*\varphi^*(\mathfrak{d})))$. This proves that the ideal quotient of contracted ideals is a contracted ideal.

(b) Let $\mathfrak{c}, \mathfrak{d} \subset A$ be ideals. We show that $\varphi_*(\mathfrak{c} + \mathfrak{d}) = \varphi_*(\mathfrak{c}) + \varphi_*(\mathfrak{d})$. We have the inclusion $\varphi(\mathfrak{c} + \mathfrak{d}) \subset \varphi_*(\mathfrak{c}) + \varphi_*(\mathfrak{d})$. Since the right hand side is an ideal we conclude that $\varphi_*(\mathfrak{c} + \mathfrak{d}) \subset \varphi_*(\mathfrak{c}) + \varphi_*(\mathfrak{d})$. Conversely we note that $\varphi_*(\mathfrak{c} + \mathfrak{d}) \supset \varphi(\mathfrak{c} + \mathfrak{d}) \supset \varphi(\mathfrak{c})$ and since the left hand side is an ideal also $\varphi_*(\mathfrak{c} + \mathfrak{d}) \supset \varphi_*(\mathfrak{c})$ and similarly $\varphi_*(\mathfrak{c} + \mathfrak{d}) \supset \varphi_*(\mathfrak{d})$. Since the sum of two ideals is the smallest ideal containing both ideals, we conclude that

$\varphi_*(\mathfrak{c}+\mathfrak{d}) \supset \varphi_*(\mathfrak{c})+\varphi_*(\mathfrak{d})$. This shows that the sum of two extended ideals is again an extended ideal.

We show that $\varphi_*(\mathfrak{c}\mathfrak{d}) = \varphi_*(\mathfrak{c})\varphi_*(\mathfrak{d})$. By definition of the product of ideals we have $\varphi(\mathfrak{c}\mathfrak{d}) \subset \varphi_*(\mathfrak{c})\varphi_*(\mathfrak{d})$ and thus $\varphi_*(\mathfrak{c}\mathfrak{d}) \subset \varphi_*(\mathfrak{c})\varphi_*(\mathfrak{d})$. Conversely every element in $\varphi_*(\mathfrak{c})\varphi_*(\mathfrak{d})$ can be written as a finite sum of products $ab$ with $a \in \varphi_*(\mathfrak{c})$ and $b \in \varphi_*(\mathfrak{d})$. These elements can again be expressed as finite linear combinations of elements in $\varphi(\mathfrak{c})$ and $\varphi(\mathfrak{d})$, respectively. Multiplying all out and using that $\varphi$ is a homomorphism we get a finite linear combination of elements $\varphi(cd)$ with $c \in \mathfrak{c}$ and $d \in \mathfrak{d}$. Hence it is contained in $\varphi_*(\mathfrak{c}\mathfrak{d})$. This proves that the product of extended ideals is an extended ideal.

2. Let $A$ be a ring and $\mathfrak{a} \subset A$ be an ideal that is contained in the Jacobson radical of $A$. Let $M, N$ be $A$-modules, where $N$ is finitely generated, and let $\varphi : M \to N$ be an $A$-module homomorphism. Consider the induced homomorphism

$$\varphi_{\mathfrak{a}} : M/\mathfrak{a}M \to N/\mathfrak{a}N$$

Prove that if $\varphi_{\mathfrak{a}}$ is surjective, then $\varphi$ is surjective.

*Solution*: We first show that $\mathfrak{a}\left(N/\varphi(M)\right) = N/\varphi(M)$. Clearly, the left hand side is contained in the right hand side. Conversely, let $n \in N$. Because $\varphi_{\mathfrak{a}}$ is surjective, there is an element $m \in M$ such that $\varphi(m) - n \in \mathfrak{a}N$. Choose $a \in \mathfrak{a}$ and $n' \in N$ such that $\varphi(m) - n = an'$. Hence $n + an' \in \varphi(M)$. We conclude that $[n] = [an'] = a[n']$ in $N/\varphi(M)$ and thus the equality $\mathfrak{a}\left(N/\varphi(M)\right) = N/\varphi(M)$. By the fact that a quotient module of a finitely generated module is still finitely generated we can use Nakayama's lemma to conclude that $N/\varphi(M) = 0$ and hence $\varphi$ is surjective.

3. Let $k$ be a field and $0 \to M_0 \to \cdots \to M_n \to 0$ be an exact sequence of finite dimensional $k$-vector spaces and $k$-linear maps. Prove that

$$\sum_{i=0}^{n}(-1)^i \dim_k(M_i) = 0$$

*Solution*: Denote $d_i$ for the map $d_i : M_i \to M_{i+1}$, where we denote $M_k = 0$ for $k > n$. The long exact sequence splits into short exact sequences $0 \to \ker(d_i) \to M_i \to \operatorname{im}(d_i) \to 0$. By using the dimension formula for linear maps of finite dimensional vector spaces and the fact $\ker(d_{i+1}) = \operatorname{im}(d_i)$ we conclude that $\dim_k(\ker(d_i)) + \dim_k(\ker(d_{i+1})) = \dim_k(M_i)$. Thus

$$\sum_{i=0}^{n}(-1)^i \dim_k(M_i) = \dim_k(\ker(d_0)) + (-1)^n \dim_k(\ker(d_{n+1})) = 0.$$

4. Prove the 4-Lemma by diagram chasing: If the rows of the commutative diagram of $A$-modules and $A$-module homomorphisms

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d, "\alpha"] & M_2 \arrow[r] \arrow[d, "\beta"] & M_3 \arrow[r] \arrow[d, "\gamma"] & M_4 \arrow[d, "\delta"] \\ M_1' \arrow[r] & M_2' \arrow[r] & M_3' \arrow[r] & M_4' \end{tikzcd}$$

are exact, then the following holds:

(a) If $\alpha$ is surjective, and $\beta$ and $\delta$ are injective, then $\gamma$ is injective;

(b) if $\delta$ is injective, and $\alpha$ and $\gamma$ are surjective, then $\beta$ is surjective.

*Solution*: We will denote $d_i : M_i \to M_{i+1}$ and $d_i' : M_i' \to M_{i+1}'$.

(a) Let $m_3 \in M_3$ with $\gamma(m_3) = 0$. By commutativity of the right square and injectivity of $\delta$ we conclude that $d_3(m_3) = 0$. Thus there is an element $m_2 \in M_2$ such that $d_2(m_2) = m_3$. Consider the element $\beta(m_2)$. We have $d_2'(\beta(m_2)) = \gamma(m_3) = 0$. We conclude that there is an element $m_1' \in M_1'$ such that $d_1'(m_1') = \beta(m_2)$. Since $\alpha$ is surjective, there is an element $m_1 \in M_1$ such that $\alpha(m_1) = m_1'$. By injectivity of $\beta$ and using $\beta(d_1(m_1)) = d_1'(\alpha(m_1)) = \beta(m_2)$ we conclude that $d_1(m_1) = m_2$. Hence $m_3 = d_2(m_2) = d_2(d_1(m_1)) = 0$. This proves that $\gamma$ is injective.

(b) Let $m_2' \in M_2'$ and look at $d_2'(m_2')$. Since $\gamma$ is surjective there is an element $m_3 \in M_3$ such that $\gamma(m_3) = d_2'(m_2')$. We have $\delta(d_3(m_3)) = d_3'(\gamma(m_3)) = d_3'(d_2'(m_2')) = 0$. Since $\delta$ is injective this implies that $d_3(m_3) = 0$. Thus there is an element $m_2 \in M_2$ such that $d_2(m_2) = m_3$. By commutativity of the middle square we conclude that $d_2'(\beta(m_2) - m_2') = 0$. Hence there is an element $a' \in M_1'$ such that $d_1'(a') = \beta(m_2) - m_2'$. Since $\alpha$ is surjective we can lift this to an element $a \in M_1$ such that $\alpha(a) = a'$. Finally we conclude that $\beta(m_2 - d_1(a)) = d_1'(a') + m_2' - d_1'(\alpha(a)) = m_2'$, which proves that $\beta$ is surjective.

5. Prove the $3 \times 3$-lemma: If

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M_1' \arrow[r] \arrow[d] & M_2' \arrow[r] \arrow[d] & M_3' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] \arrow[d] & M_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1'' \arrow[r] \arrow[d] & M_2'' \arrow[r] \arrow[d] & M_3'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a commutative diagram of $A$-modules and $A$-module homomorphisms, and all columns and the middle row are exact, then the top row is exact if and only if the bottom row is exact.

*Solution*: This follows directly from the snake lemma: if the bottom row is exact, look at the morphisms $M_i \to M_i''$ and use the snake lemma. If the top row is exact, use the snake lemma for the morphisms $M_i' \to M_i$.

6. In this exercise, we generalize the notion of an affine variety introduced in the lecture. Let $A$ be a ring. We denote by $\mathrm{spec}(A)$ the set of all prime ideals of $A$. For a subset $S \subset A$ define

$$V(S) := \{\mathfrak{p} \in \mathrm{spec}(A) \mid S \subset \mathfrak{p}\}$$

Show that:

(a) If $\mathfrak{a} \subset A$ is the ideal generated by $S$, then $V(S) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

(b) $V(0) = \mathrm{spec}(A)$ and $V(1) = \varnothing$.

(c) For a family of subsets $(S_i)_{i \in I} \subset A$ we have $V(\bigcup_{i \in I} S_i) = \bigcap_{i \in I} V(S_i)$.

(d) For finitely many ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_n \subset A$ we have $V(\bigcap_{i=1}^n \mathfrak{a}_i) = \bigcup_{i=1}^n V(\mathfrak{a}_i)$.

This shows that the subsets $(V(S))_{S \subset A}$ form the closed sets of a topology on $\mathrm{spec}(A)$, called *Zariski* topology. We call the topological space $\mathrm{spec}(A)$ the *(prime) spectrum* of $A$.

*Solution*:

(a) Every prime ideal that containes $S$ also contains $\mathfrak{a}$ and vice versa. Thus $V(S) = V(\mathfrak{a})$. The radical of $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$. Thus every prime ideal that contains $\mathfrak{a}$ contains $r(\mathfrak{a})$ and the converse is clear. Hence $V(\mathfrak{a}) = V(r(\mathfrak{a}))$.

(b) Every prime ideal contains the zero ideal and no prime ideal contains the unit ideal, hence $V(0) = \mathrm{spec}(A)$ and $V(1) = \varnothing$.

(c) This follows by set theory.

(d) We use a proposition from the lecture. If a prime ideal $\mathfrak{p}$ contains $\bigcap_{i=1}^n \mathfrak{a}_i$, then it contains one of the $\mathfrak{a}_i$ by the proposition. Thus $V(\bigcap_{i=1}^n \mathfrak{a}_i) \subset \bigcup_{i=1}^n V(\mathfrak{a}_i)$. The converse is true by set theory.