## Solutions Sheet 3

## TENSOR PRODUCT, MODULES, SPECTRUM OF A RING

1. Let A be a local ring and M, N two finitely generated A-modules. Prove that  $M \otimes_A N = 0$  implies M = 0 or N = 0. Give an example of modules over a non-local ring which do not have this property.

Solution: Denote by  $\mathfrak{m}$  the maximal ideal of A and by  $k := A/\mathfrak{m}A$  the residue field. Assume that  $M \otimes_A N = 0$ . Naturally  $M/\mathfrak{m}M$  and  $N/\mathfrak{m}N$  are not only A-modules but also k-vector spaces. First we prove that  $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) = 0$ . Look at the surjective map  $M \twoheadrightarrow M/\mathfrak{m}M$ . By right exactness of the tensor product we conclude that  $M \otimes_A N \twoheadrightarrow M/\mathfrak{m}M \otimes_A N$  is surjective. Similarly we find that  $(M/\mathfrak{m}M) \otimes_A N \twoheadrightarrow (M/\mathfrak{m}M) \otimes_A (N/\mathfrak{m}N)$  is surjective. Hence the composite map is surjective, which proves that  $(M/\mathfrak{m}M) \otimes_A (N/\mathfrak{m}N) = 0$ . We note that the tensor map  $(M/\mathfrak{m}M) \times (N/\mathfrak{m}N) \to (M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N)$  is k-bilinear and thus in particular A-bilinear. Hence it factors through  $(M/\mathfrak{m}M) \otimes_A (N/\mathfrak{m}N)$ by the universal property. We conclude that the tensor map is zero and thus  $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) = 0$ . Now by the dimension formula of tensor products of vector spaces (or by considering an explicit basis) we conclude that  $M/\mathfrak{m}M = 0$ or  $N/\mathfrak{m}N = 0$  and thus  $M = \mathfrak{m}M$  or  $N = \mathfrak{m}N$ . By Nakayama's Lemma it follows that M = 0 or N = 0.

An example where this is not true over a non-local ring is  $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$ . This is zero because for every elementary tensor  $a \otimes b$  we have  $a \otimes b = (3a) \otimes b = a \otimes (3b) = a \otimes 0 = 0$ .

- 2. Let A be a ring. Prove the following:
  - (a) If M and N are flat A-modules, then so is  $M \otimes_A N$ .
  - (b) If B is a flat A-algebra and M a flat B-module, then M is flat as an A-module.

## Solution:

(a) Let  $L \hookrightarrow L'$  be an injective homomorphism of A-modules. Since M and N are flat and using a proposition from the lecture we conclude that the induced homomorphism  $L \otimes_A M \to L' \otimes_A M$  is injective. Using this proposition again we conclude that  $(L \otimes_A M) \otimes_A N \to (L' \otimes_A M) \otimes_A N$  is injective. By associativity of the tensor product we conclude that  $L \otimes_A (M \otimes_A N) \to$  $L' \otimes_A (M \otimes_A N)$  is injective and hence  $M \otimes_A N$  is flat, by using the proposition again. (b) Let  $L \hookrightarrow L'$  be an injective homomorphism of A-modules. We use the Aisomorphism  $(L \otimes_A B) \otimes_B M \cong L \otimes_A M$  given by  $\ell \otimes b \otimes m \mapsto \ell \otimes (bm)$ and the analogue for L'. That this is indeed well-defined can be checked using the universal property of the tensor product in the following way: for every element  $m \in M$  we have an A-bilinear map  $L \times B \to L \otimes_A M$  given by  $(l, b) \mapsto l \otimes (bm)$ . Hence it factors through the tensor product  $L \otimes_A B$ . Varying  $m \in M$  we get a homomorphism  $(L \otimes_A B) \times M \to L \otimes_A M$  which is not only A-bilinear, but also B-bilinear. It thus factors through the tensor product  $(L \otimes_A B) \otimes_B M$ . Conversely the map  $L \times M \to (L \otimes_A B) \otimes_B M$ given by  $(l, m) \mapsto (l \otimes 1) \otimes m$  is A-bilinear and thus factors through the tensor product  $L \otimes_A M$ . It is not hard to see that this provides an inverse for the mentioned map and thus we have an isomorphism.

We get a commutative diagram

$$L \otimes_A M \xrightarrow{} L' \otimes_A M$$

$$\downarrow \cong \qquad \qquad \downarrow \cong$$

$$(L \otimes_A B) \otimes_B M \xrightarrow{} (L' \otimes_A B) \otimes_B M$$

The A-homomorphism at the bottom is injective because of the flatness of B as an A-module and the flatness of M as a B-module. Hence the upper A-homomorphism is injective. By the proposition from the lecture this implies that M is flat as an A-module.

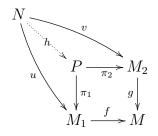
3. Let A be a ring. Consider a short exact sequence of A-modules and homomorphisms  $0 \to M' \to M \to M'' \to 0$ . Prove that if M' and M'' are finitely generated, then so is M.

Solution: Let  $m'_1, \ldots, m'_r \in M'$  and  $n''_1, \ldots, n''_s \in M''$  be generators of the respective A-modules. Denote by  $m_1, \ldots, m_r \in M$  the images of  $m'_1, \ldots, m'_r$  in M and by  $n_1, \ldots, n_s \in M$  lifts of  $n''_1, \ldots, n''_s$  in M. We claim that  $m_1, \ldots, m_r, n_1, \ldots, n_s$  generate M. Let  $a \in M$ . By assumption its image  $a'' \in M''$  can be written as  $\sum_{i=1}^s \alpha_i n''_i$  for some coefficients  $\alpha_1, \ldots, \alpha_s \in A$ . But then  $a - \sum_{i=1}^s \alpha_i n_i$  is in the kernel of the map  $M \to M''$  and thus is the image of an element  $b \in M'$ . By assumption  $b = \sum_{j=1}^r \beta_j m'_j$  for some coefficients  $\beta_1, \ldots, \beta_r \in A$ . We conclude that  $a = \sum_{i=1}^s \alpha_i n_i + \sum_{j=1}^r \beta_j m_j$ . This proves the claim.

4. Let A be a ring. Prove that for any three A-modules  $M_1, M_2, M$  and homomorphisms  $M_1 \xrightarrow{f} M \xleftarrow{g} M_2$  there exists an A-module P and homomorphisms  $M_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_2} M_2$  such that the diagram

$$\begin{array}{c} P \xrightarrow{\pi_2} M_2 \\ \downarrow^{\pi_1} & \downarrow^g \\ M_1 \xrightarrow{f} M \end{array}$$

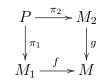
commutes and with the following universal property: for any A-module N and homomorphisms  $M_1 \stackrel{u}{\leftarrow} N \stackrel{v}{\rightarrow} M_2$  such that  $f \circ u = g \circ v$  there exists a unique homomorphism  $h: N \to P$  making the whole diagram commute:



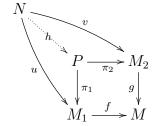
Finally, show that P is unique up to a unique isomorphism.

[Hint: Look at a submodule of  $M_1 \times M_2$ .]

Solution: We define  $P := \{(m_1, m_2) \in M_1 \oplus M_2 \mid f(m_1) = g(m_2)\}$ . Since it is the kernel of the homomorphism  $M_1 \oplus M_2 \to M$  given by  $(m_1, m_2) \mapsto f(m_1) - g(m_2)$  it is an A-module. We define  $\pi_1 : P \to M_1$  and  $\pi_2 : P \to M_2$  to be the respective projections of  $M_1 \oplus M_2$  restricted to P. By the very definition of P the diagram

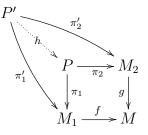


commutes. We show that  $(P, \pi_1, \pi_2)$  satisfies the required universal property. Let N be an A-module and let  $u : N \to M_1$  and  $v : N \to M_2$  be homomorphisms such that  $f \circ u = g \circ v$ . We construct a homomorphism  $h : N \to P$  by h(n) := (u(n), v(n)). By the requirement  $f \circ u = g \circ v$  this is indeed well-defined. And it is a homomorphism, since u and v are homomorphisms. Furthermore, we see that the diagram

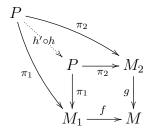


commutes. The uniqueness of h follows directly from the construction. Now let  $(P', \pi'_1, \pi'_2)$  be another module with the same universal property. Then by the universal property of P we have a unique homomorphism  $h: P' \to P$  such that

the diagram



commutes. Using the universal property for P' we analogously get a homomorphism  $h': P \to P'$ . Composing those homomorphisms we get a homomorphism  $h' \circ h: P \to P$  such that the diagram



commutes. By the uniqueness statement of the universal property of P we conclude that  $h' \circ h$  is the identity. Hence h is a unique isomorphism.

- 5. Let A be a ring. Recall the definition of the prime spectrum of a ring from exercise sheet 2. For every element  $f \in A$  denote D(f) for the open complement of V((f)) in spec(A). Show that these sets form a basis of open sets for the Zariski topology on spec(A). Furthermore, prove:
  - (a)  $\forall f, g \in A$  we have  $D(f) \cap D(g) = D(fg)$
  - (b)  $D(f) = \emptyset$  if and only if f is nilpotent
  - (c)  $D(f) = \operatorname{spec}(A)$  if and only if f is a unit
  - (d)  $\operatorname{spec}(A)$  is quasicompact

These open sets are called *basic open sets* of spec(A).

Solution: By definition we can write  $D(f) = \{\mathfrak{p} \in \operatorname{spec}(A) \mid f \notin \mathfrak{p}\}$  for  $f \in A$ . We need to show that the basic open sets cover  $\operatorname{spec}(A)$  and that for every two basic open sets  $B_1, B_2$  and every point  $x \in B_1 \cap B_2$  there is a basic open set  $B_3$ such that  $x \in B_3 \subset B_1 \cap B_2$ . The first property is true because of  $D(1) = \operatorname{spec}(A)$ . For the second one let D(f), D(g) with  $f, g \in A$  be basic open sets. By (a) below we have  $D(f) \cap D(g) = D(fg)$  and thus the second property of being a basis for the topology is also satisfied. Hence the basic open sets form indeed a basis for the topology.

- (a) For every prime ideal  $\mathfrak{p} \subset A$  we have  $fg \notin \mathfrak{p}$  if and only if  $f \notin \mathfrak{p}$  and  $g \notin \mathfrak{p}$ . Thus the equality.
- (b) Since the nilradical is the intersection of all prime ideals, every prime ideal contains all nilpotent elements. Thus  $D(f) = \emptyset$  for all nilpotent elements  $f \in A$ .
- (c) By definition a prime ideal cannot contain a unit, hence  $D(f) = \operatorname{spec}(A)$  for every unit  $f \in A$ .
- (d) Since the basic open sets form a basis of the topology it is enough to consider covers by basic open sets. Let  $(f_i)_{i \in I} \in A$  be elements such that  $\bigcup_{i \in I} D(f_i) = \operatorname{spec}(A)$ . By taking the complement and using de Morgans law we get  $\bigcap_{i \in I} V((f_i)) = \emptyset$ . By exercise sheet 2, exercise 6(c) and 6(a) we conclude that  $V(\sum_{i \in I} (f_i)) = \emptyset$ . Hence the ideal  $\sum_{i \in I} (f_i)$  must contain 1. Thus there is a finite subset  $J \subset I$  such that  $\sum_{j \in J} D(f_j) = \operatorname{spec}(A)$ . Hence the ideal by the above argument in reverse, we have  $\bigcup_{j \in J} D(f_j) = \operatorname{spec}(A)$ . Hence the finite subcover already covers  $\operatorname{spec}(A)$ .