Solutions Sheet 3

Tensor Product, Modules, Spectrum of a Ring

1. Let A be a local ring and M, N two finitely generated A-modules. Prove that $M \otimes_A N = 0$ implies $M = 0$ or $N = 0$. Give an example of modules over a non-local ring which do not have this property.

   Solution: Denote by $\mathfrak{m}$ the maximal ideal of A and by $k := A/\mathfrak{m}A$ the residue field. Assume that $M \otimes_A N = 0$. Naturally $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are not only A-modules but also k-vector spaces. First we prove that $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) = 0$. Look at the surjective map $M \twoheadrightarrow M/\mathfrak{m}M$. By right exactness of the tensor product we conclude that $M \otimes_A N \twoheadrightarrow M/\mathfrak{m}M \otimes_A N$ is surjective. Similarly we find that $(M/\mathfrak{m}M) \otimes_A N \twoheadrightarrow (M/\mathfrak{m}M) \otimes_A (N/\mathfrak{m}N)$ is surjective. Hence the composite map is surjective, which proves that $(M/\mathfrak{m}M) \otimes_A (N/\mathfrak{m}N) = 0$. We note that the tensor map $(M/\mathfrak{m}M) \times (N/\mathfrak{m}N) \to (M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N)$ is k-bilinear and thus in particular A-bilinear. Hence it factors through $(M/\mathfrak{m}M) \otimes_A (N/\mathfrak{m}N)$ by the universal property. We conclude that the tensor map is zero and thus $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) = 0$. Now by the dimension formula of tensor products of vector spaces (or by considering an explicit basis) we conclude that $M/\mathfrak{m}M = 0$ or $N/\mathfrak{m}N = 0$ and thus $M = \mathfrak{m}M$ or $N = \mathfrak{m}N$. By Nakayama's Lemma it follows that $M = 0$ or $N = 0$.

   An example where this is not true over a non-local ring is $\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$. This is zero because for every elementary tensor $a \otimes b$ we have $a \otimes b = (3a) \otimes b = a \otimes (3b) = a \otimes 0 = 0$.

2. Let A be a ring. Prove the following:

   (a) If M and N are flat A-modules, then so is $M \otimes_A N$.

   (b) If B is a flat A-algebra and M a flat B-module, then M is flat as an A-module.

   Solution:

   (a) Let $L \hookrightarrow L'$ be an injective homomorphism of A-modules. Since M and N are flat and using a proposition from the lecture we conclude that the induced homomorphism $L \otimes_A M \to L' \otimes_A M$ is injective. Using this proposition again we conclude that $(L \otimes_A M) \otimes_A N \to (L' \otimes_A M) \otimes_A N$ is injective. By associativity of the tensor product we conclude that $L \otimes_A (M \otimes_A N) \to L' \otimes_A (M \otimes_A N)$ is injective and hence $M \otimes_A N$ is flat, by using the proposition again.

(b) Let $L \hookrightarrow L'$ be an injective homomorphism of A-modules. We use the A-isomorphism $(L \otimes_A B) \otimes_B M \cong L \otimes_A M$ given by $\ell \otimes b \otimes m \mapsto \ell \otimes (bm)$ and the analogue for L'. That this is indeed well-defined can be checked using the universal property of the tensor product in the following way: for every element $m \in M$ we have an A-bilinear map $L \times B \to L \otimes_A M$ given by $(l, b) \mapsto l \otimes (bm)$. Hence it factors through the tensor product $L \otimes_A B$. Varying $m \in M$ we get a homomorphism $(L \otimes_A B) \times M \to L \otimes_A M$ which is not only A-bilinear, but also B-bilinear. It thus factors through the tensor product $(L \otimes_A B) \otimes_B M$. Conversely the map $L \times M \to (L \otimes_A B) \otimes_B M$ given by $(l, m) \mapsto (l \otimes 1) \otimes m$ is A-bilinear and thus factors through the tensor product $L \otimes_A M$. It is not hard to see that this provides an inverse for the mentioned map and thus we have an isomorphism.

We get a commutative diagram

$$\begin{tikzcd} L \otimes_A M \arrow[r] \arrow[d, "\cong"] & L' \otimes_A M \arrow[d, "\cong"] \\ (L \otimes_A B) \otimes_B M \arrow[r] & (L' \otimes_A B) \otimes_B M \end{tikzcd}$$

The A-homomorphism at the bottom is injective because of the flatness of B as an A-module and the flatness of M as a B-module. Hence the upper A-homomorphism is injective. By the proposition from the lecture this implies that M is flat as an A-module.

3. Let A be a ring. Consider a short exact sequence of A-modules and homomorphisms $0 \to M' \to M \to M'' \to 0$. Prove that if M' and M'' are finitely generated, then so is M.

Solution: Let $m'_1, \ldots, m'_r \in M'$ and $n''_1, \ldots, n''_s \in M''$ be generators of the respective A-modules. Denote by $m_1, \ldots, m_r \in M$ the images of $m'_1, \ldots, m'_r$ in M and by $n_1, \ldots, n_s \in M$ lifts of $n''_1, \ldots, n''_s$ in M. We claim that $m_1, \ldots, m_r, n_1, \ldots, n_s$ generate M. Let $a \in M$. By assumption its image $a'' \in M''$ can be written as $\sum_{i=1}^s \alpha_i n''_i$ for some coefficients $\alpha_1, \ldots, \alpha_s \in A$. But then $a - \sum_{i=1}^s \alpha_i n_i$ is in the kernel of the map $M \to M''$ and thus is the image of an element $b \in M'$. By assumption $b = \sum_{j=1}^r \beta_j m'_j$ for some coefficients $\beta_1, \ldots, \beta_r \in A$. We conclude that $a = \sum_{i=1}^s \alpha_i n_i + \sum_{j=1}^r \beta_j m_j$. This proves the claim.

4. Let A be a ring. Prove that for any three A-modules M_1, M_2, M and homomorphisms $M_1 \xrightarrow{f} M \xleftarrow{g} M_2$ there exists an A-module P and homomorphisms $M_1 \xleftarrow{\pi_1} P \xrightarrow{\pi_2} M_2$ such that the diagram

$$\begin{tikzcd} P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & M_2 \arrow[d, "g"] \\ M_1 \arrow[r, "f"] & M \end{tikzcd}$$

commutes and with the following universal property: for any A-module N and homomorphisms $M_1 \xleftarrow{u} N \xrightarrow{v} M_2$ such that $f \circ u = g \circ v$ there exists a unique homomorphism $h : N \to P$ making the whole diagram commute:

$$\begin{tikzcd}
N \arrow[dr, dotted, "h"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & M_2 \arrow[d, "g"'] \\
& M_1 \arrow[r, "f"] & M
\end{tikzcd}$$

Finally, show that P is unique up to a unique isomorphism.

[Hint: Look at a submodule of $M_1 \times M_2$.]

Solution: We define $P := \{(m_1, m_2) \in M_1 \oplus M_2 \mid f(m_1) = g(m_2)\}$. Since it is the kernel of the homomorphism $M_1 \oplus M_2 \to M$ given by $(m_1, m_2) \mapsto f(m_1) - g(m_2)$ it is an A-module. We define $\pi_1 : P \to M_1$ and $\pi_2 : P \to M_2$ to be the respective projections of $M_1 \oplus M_2$ restricted to P. By the very definition of P the diagram

$$\begin{tikzcd}
P \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & M_2 \arrow[d, "g"] \\
M_1 \arrow[r, "f"] & M
\end{tikzcd}$$

commutes. We show that (P, π_1, π_2) satisfies the required universal property. Let N be an A-module and let $u : N \to M_1$ and $v : N \to M_2$ be homomorphisms such that $f \circ u = g \circ v$. We construct a homomorphism $h : N \to P$ by $h(n) := (u(n), v(n))$. By the requirement $f \circ u = g \circ v$ this is indeed well-defined. And it is a homomorphism, since u and v are homomorphisms. Furthermore, we see that the diagram

$$\begin{tikzcd}
N \arrow[dr, dotted, "h"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] & & \\
& P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & M_2 \arrow[d, "g"'] \\
& M_1 \arrow[r, "f"] & M
\end{tikzcd}$$

commutes. The uniqueness of h follows directly from the construction. Now let (P', π_1', π_2') be another module with the same universal property. Then by the universal property of P we have a unique homomorphism $h : P' \to P$ such that the diagram

$$\begin{tikzcd}
P' \arrow[dr, dotted, "h"] \arrow[drr, bend left, "\pi_2'"] \arrow[ddr, bend right, "\pi_1'"'] & & \\
& P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & M_2 \arrow[d, "g"'] \\
& M_1 \arrow[r, "f"] & M
\end{tikzcd}$$

commutes. Using the universal property for P' we analogously get a homomorphism $h' : P \to P'$. Composing those homomorphisms we get a homomorphism $h' \circ h : P \to P$ such that the diagram

$$\begin{tikzcd}
P \arrow[dr, dotted, "h' \circ h"] \arrow[drr, bend left, "\pi_2"] \arrow[ddr, bend right, "\pi_1"'] & & \\
& P \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & M_2 \arrow[d, "g"'] \\
& M_1 \arrow[r, "f"] & M
\end{tikzcd}$$

commutes. By the uniqueness statement of the universal property of P we conclude that $h' \circ h$ is the identity. Hence h is a unique isomorphism.

5. Let A be a ring. Recall the definition of the prime spectrum of a ring from exercise sheet 2. For every element $f \in A$ denote $D(f)$ for the open complement of $V((f))$ in $\operatorname{spec}(A)$. Show that these sets form a basis of open sets for the Zariski topology on $\operatorname{spec}(A)$. Furthermore, prove:

   (a) $\forall f, g \in A$ we have $D(f) \cap D(g) = D(fg)$

   (b) $D(f) = \varnothing$ if and only if f is nilpotent

   (c) $D(f) = \operatorname{spec}(A)$ if and only if f is a unit

   (d) $\operatorname{spec}(A)$ is quasicompact

   These open sets are called *basic open sets* of $\operatorname{spec}(A)$.

   Solution: By definition we can write $D(f) = \{\mathfrak{p} \in \operatorname{spec}(A) \mid f \notin \mathfrak{p}\}$ for $f \in A$. We need to show that the basic open sets cover $\operatorname{spec}(A)$ and that for every two basic open sets B_1, B_2 and every point $x \in B_1 \cap B_2$ there is a basic open set B_3 such that $x \in B_3 \subset B_1 \cap B_2$. The first property is true because of $D(1) = \operatorname{spec}(A)$. For the second one let $D(f), D(g)$ with $f, g \in A$ be basic open sets. By (a) below we have $D(f) \cap D(g) = D(fg)$ and thus the second property of being a basis for the topology is also satisfied. Hence the basic open sets form indeed a basis for the topology.

(a) For every prime ideal $\mathfrak{p} \subset A$ we have $fg \notin \mathfrak{p}$ if and only if $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$. Thus the equality.

(b) Since the nilradical is the intersection of all prime ideals, every prime ideal contains all nilpotent elements. Thus $D(f) = \varnothing$ for all nilpotent elements $f \in A$.

(c) By definition a prime ideal cannot contain a unit, hence $D(f) = \operatorname{spec}(A)$ for every unit $f \in A$.

(d) Since the basic open sets form a basis of the topology it is enough to consider covers by basic open sets. Let $(f_i)_{i\in I} \in A$ be elements such that $\bigcup_{i\in I} D(f_i) = \operatorname{spec}(A)$. By taking the complement and using de Morgans law we get $\bigcap_{i\in I} V((f_i)) = \varnothing$. By exercise sheet 2, exercise 6(c) and 6(a) we conclude that $V(\sum_{i\in I}(f_i)) = \varnothing$. Hence the ideal $\sum_{i\in I}(f_i)$ must contain 1. Thus there is a finite subset $J \subset I$ such that $\sum_{j\in J}(f_j)$ contains 1. But then by the above argument in reverse, we have $\bigcup_{j\in J} D(f_j) = \operatorname{spec}(A)$. Hence the finite subcover already covers $\operatorname{spec}(A)$.