Commutative Algebra

HS 2017

Solutions Sheet 4

Localisation, Splitting Lemma, Irreducible Variety

1. Let A be a ring ~~reduced ring (i.e. without any nonzero nilpotent elements)~~. Let M be a finitely generated A-module and let $f : M \to M$ be a surjective module homomorphism. Then f is also injective.

   Remark: The intended proof did not work, so we give a general proof which does not need A to be reduced. Many apologies for this inconvenience!

   Solution:

   One variant of Nakayama says that if N is a finitely generated A-module and $\mathfrak{a} \subset A$ an ideal such that $\mathfrak{a}N = N$, then there is an element $x \in 1 + \mathfrak{a}$ such that $xN = 0$. We use this as follows: We consider M as $A[X]$-module, where X acts as f on M, i.e. for any $p(X) \in A[X]$ and $m \in M$ we have $p(X) \cdot m = p(f)(m)$. Since f is surjective, for the ideal $\mathfrak{a} := (X)$ we have $\mathfrak{a}M = M$. Hence there is an element $a \in 1+\mathfrak{a}$ such that $aM = 0$, which implies that there is a polynomial $q(X) \in A[X]$ such that $1 + q(X)X = a$. We conclude that for every element $b \in \ker(f)$ we have $b = (1 + q(X)X)b = ab = 0$. Hence f is injective.

2. Let A be a ring such that every localisation $A_\mathfrak{p}$ of A with respect to a prime ideal $\mathfrak{p} \subset A$ has no nonzero nilpotent elements. Prove that A has no nonzero nilpotent elements. Is the same true for zero-divisors?

   Solution: We have $0 = \mathfrak{nil}(A_\mathfrak{p}) = \mathfrak{nil}(A)_\mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset A$. Since being zero is a local property of an A-module, we conclude that $\mathfrak{nil}(A) = 0$. For zero-divisors this is not true as the following example shows: consider $\mathbb{Q} \times \mathbb{Q}$ as ring. It certainly has zero-divisors and the only prime ideals are $\{0\} \times \mathbb{Q}$ and $\mathbb{Q} \times \{0\}$. Denote by R the localisation at $\{0\} \times \mathbb{Q}$. Assume that $[(a, b) : (0, c)]$ is a zero-divisor in R. Then there is an element $[(d, e) : (0, f)]$ in R such that their product $[(ad, eb) : (0, cf)]$ is zero, so by definition of the localisation there is an element $(g, h) \notin \{0\} \times \mathbb{Q}$ such that $(adg, ebh) = (ad, eb)(g, h) = 0$. Since g is non-zero, either a or d is zero. But then $(a, b)(1, 0) = 0$ or $(d, e)(1, 0) = 0$ and so one of the elements $[(a, b) : (0, c)]$ or $[(d, e) : (0, f)]$ must be zero in R already and thus is no non-zero zero-divisor.

3. Let A be a ring. Let T, S be two multiplicatively closed subsets and let U be the image of T in $S^{-1}A$. Prove that $(ST)^{-1}A$ is isomorphic to $U^{-1}S^{-1}A$.

   Solution: Consider the canonical map $f : A \to (ST)^{-1}A$. Since $S \subset ST$, the elements of the subset $f(S) \subset (ST)^{-1}A$ are invertible and thus f factors through a unique homomorphism $g : S^{-1}A \to (ST)^{-1}A$ by the universal property. But as $T \subset ST$, we see that the elements of $g(U) = f(T) \subset (ST)^{-1}A$ are invertible and by the universal property g factors through a unique homomorphism $h : U^{-1}S^{-1}A \to (ST)^{-1}A$. Conversely consider the map $f' : A \to U^{-1}S^{-1}A$. We see that the elements of $f'(ST)$ are invertible and thus f' factors through a unique homomorphism $h' : (ST)^{-1}A \to U^{-1}S^{-1}A$. By using the uniqueness we conclude that $h \circ h'$ and $h' \circ h$ are both the respective identity homomorphisms. Thus h is an isomorphism with inverse h'.

4. Let A be an integral domain and M an A-module. Prove that the following are equivalent:

   (a) M is torsion-free.

   (b) $M_\mathfrak{p}$ is torsion-free for all prime ideals $\mathfrak{p} \subset A$.

   (c) $M_\mathfrak{m}$ is torsion-free for all maximal ideals $\mathfrak{m} \subset A$.

   Solution: "(a)$\Rightarrow$(b)": Assume that there is a prime ideal $\mathfrak{p} \subset A$ such that $M_\mathfrak{p}$ has a torsion element $[m, s] \in M_\mathfrak{p}$, where $m \in M$ is non-zero and $s \notin \mathfrak{p}$. Thus there is a non-zero element $a \in A$ such that $a[m, s] = [am, s] = 0$. By definition of localisation there is an element $r \notin \mathfrak{p}$ such that $ram = 0$. Since A is an integral domain we conclude that $am = 0$ and hence M has torsion.

   "(b)$\Rightarrow$(c)": Immediate.

   "(c)$\Rightarrow$(a)": Let $m \in M$ be a non-zero torsion element. Consider the annihilator $\mathrm{Ann}(m) := \{a \in A \mid am = 0\}$ of m. It is an ideal which clearly does not contain 1. Hence there is a maximal ideal $\mathfrak{m} \subset A$ containing $\mathrm{Ann}(m)$. The element $[m, 1]$ is non-zero in $M_\mathfrak{m}$ because there is no element $s \notin \mathfrak{m}$ such that $sm = 0$. But $[m, 1]$ is still annihilated by a non-zero element of A and hence $M_\mathfrak{m}$ has torsion.

5. (Splitting Lemma) Let A be a ring and $0 \to M' \to M \to M'' \to 0$ a short exact sequence of A-modules. The sequence is called *split* if there is an isomorphism $M \to M' \oplus M''$ such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "u"] \arrow[d, equal] & M \arrow[r, "v"] \arrow[d, "\cong"] & M'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r] & M' \oplus M'' \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

   commutes, where the homomorphisms in the lower row are the inclusion and projection respectively.

   Prove the splitting lemma, i.e. that the following are equivalent:

   (a) The short exact sequence splits.

   (b) There is a homomorphism $i : M'' \to M$ such that $v \circ i = \mathrm{id}_{M''}$.

(c) There is a homomorphism $s : M \to M'$ such that $s \circ u = \mathrm{id}_{M'}$.

Solution: Denote $\pi_1 : M' \oplus M'' \to M'$ and $\pi_2 : M' \oplus M'' \to M''$ for the respective projections, and $\varphi_1 : M' \to M' \oplus M''$ and $\varphi_2 : M'' \to M' \oplus M''$ for the respective inclusions.

"(a)$\Rightarrow$(b)": Denote $f : M \to M' \oplus M''$ for the given isomorphism. We define the homomorphism $i : M'' \to M$ to be $i := f^{-1} \circ \varphi_2$. Then $v \circ i = \pi_2 \circ f \circ f^{-1} \circ \varphi_2 = \mathrm{id}_{M''}$.

"(a)$\Rightarrow$(c)": Denote $f : M \to M' \oplus M''$ for the given isomorphism. We define the homomorphism $s : M \to M'$ to be $s := \pi_1 \circ f$. Then $s \circ u = \pi_1 \circ f \circ f^{-1} \varphi_1 = \mathrm{id}_{M'}$.

"(b)$\Rightarrow$(a)": We define a homomorphism $g : M' \oplus M'' \to M$ as $g := u + i$. Note that $(0 \oplus v) \circ g = 0 \oplus \mathrm{id}_{M''}$. Thus for any element $(m', m'') \in M' \oplus M''$ with $g(m', m'') = 0$ we conclude that $m'' = 0$. Hence $u(m') = 0$. Since u is injective we conclude that $m' = 0$. This shows that g is injective. On the other hand, let $m \in M$ and consider the element $n := m - i \circ v(m)$. This element is in the kernel of v and thus lifts to an element $n' \in M'$. We conclude that $g(n', v(m)) = u(n') + i \circ v(m) = m$ and hence g is surjective and therefore an isomorphism. The commutativity of the diagram holds by design.

"(c)$\Rightarrow$(a)": Define a homomorphism $f : M \to M' \oplus M''$ by $f := s \oplus v$. Note that $f \circ u = (\mathrm{id}_{M'} \oplus 0)$. Hence any non-zero element $m \in M$ with $f(m) = 0$ cannot lie in the image of u. But then $v(m) \neq 0$ and so f is injective. On the other hand, for every element $(m', m'') \in M' \oplus M''$ we can choose a lift $n \in M$ of $m'' \in M''$ and so we have $f(u(m') + n) = (m', m'')$. Hence f is surjective and therefore an isomorphism.

6. A topological space is called *irreducible* if it is non-empty and every two non-empty open subsets have a non-empty intersection. Prove that for $\mathrm{spec}(A)$ the following are equivalent:

   (a) $\mathrm{spec}(A)$ is irreducible.

   (b) The nilradical of A is a prime ideal.

   (c) There is a dense point $x \in \mathrm{spec}(A)$, i.e. the closure of $\{x\}$ is $\overline{\{x\}} = \mathrm{spec}(A)$.

   Remark: We call such a point as in (c) a *generic point.*

   Solution: "(a)$\Rightarrow$(b)": If the nilradical is not a prime ideal, then there are elements a, b which are not in the nilradical but such that ab is in the nilradical. Hence $D(a) \cap D(b) = D(ab) = \varnothing$ by using exercise 5 on exercise sheet 3. But a and b are both not in the nilradical and thus there are prime ideals which do not contain a or b, respectively. Thus $D(a)$ and $D(b)$ are both non-empty. Hence $\mathrm{spec}(A)$ is reducible.

   "(b)$\Rightarrow$(c)": By assumption, the nilradical is itself a point in $\mathrm{spec}(A)$. Let $V(S)$ be a closed set that contains the nilradical $\mathfrak{nil}(A)$. Thus $S \subset \mathfrak{nil}(A)$. Since $\mathfrak{nil}(A)$

is the intersection of all prime ideals, we conclude that $S \subset \mathfrak{p}$ for all prime ideals $\mathfrak{p} \subset A$ and thus $V(S) = \mathrm{spec}(A)$.

"(c)$\Rightarrow$(a)": If there exists a dense point, then every non-empty open subset has to contain it. In particular $\mathrm{spec}(A)$ is non-empty. Then the intersection of every two non-empty open subsets is non-empty because it contains the dense point.