# Solutions Sheet 5

## Noetherian Rings, Modules and Topological Spaces

1. Let $A$ be a ring. If $A[X]$ is Noetherian, is $A$ necessarily Noetherian?

   *Solution*: Yes, because $A \cong A[X]/(X)$ and a quotient of a Noetherian ring is Noetherian.

2. Let $A$ be a ring such that every localisation $A_\mathfrak{p}$ at a prime ideal $\mathfrak{p} \subset A$ is Noetherian. Is $A$ necessarily Noetherian?

   *Solution*: No. A counterexample is $R := \prod_{i\in\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$, an infinite product of copies of $\mathbb{Z}/2\mathbb{Z}$. The ring $R$ is not Noetherian, since $\left(\prod_{0\leqslant i\leqslant n} \mathbb{Z}/2\mathbb{Z}\right)_{n\in\mathbb{N}}$ is a strictly increasing chain of ideals. It is not hard to see that every element in $R$ is idempotent, i.e. for every $a \in R$ we have $a^2 = a$. Pick a prime ideal $\mathfrak{p} \subset R$. Let $a \in \mathfrak{p}$. Then clearly $1-a \notin \mathfrak{p}$ and we have $a(1-a) = a - a^2 = 0$. Thus $a = 0$ in the localisation $R_\mathfrak{p}$. But every element $b \notin \mathfrak{p}$ becomes a unit in $R_\mathfrak{p}$. We conclude that $R_\mathfrak{p}$ is a field and in particular Noetherian. This provides a counter example.

3. Which of the following rings over $\mathbb{C}$ are Noetherian?

   (a) The ring of rational functions of $z$ having no pole on the circle $|z| = 1$.

   (b) The ring of power series in $z$ with a positive radius of convergence.

   (c) The ring of power series in $z$ with an infinite radius of convergence.

   (d) The ring of polynomials in $z$ whose first $k$ derivatives vanish at the origin, where $k$ is a fixed non-negative integer.

   (e) The ring of polynomials in $z, w$ whose partial derivatives with respect to $w$ vanish for $z = 0$.

   *Solution*: (Thanks to JJ for these solutions)

   (a) It is Noetherian: Define

   $$S := \mathbb{C}[z] \setminus \bigcup_{\substack{a\in\mathbb{C} \\ |a|=1}} (z-a)$$

   Then $S$ is a multiplicative subset of $\mathbb{C}[z]$ and $S^{-1}\mathbb{C}[z]$ is the ring of rational functions having no pole on the unit circle. Since $\mathbb{C}$ is a field and thus Noetherian, the Hilbert Basis Theorem states that $\mathbb{C}[z]$ is Noetherian. Further, the Noetherian property is preserved by localization, hence this ring is Noetherian.

(b) It is Noetherian: We can identify the ring of power series with positive radius of convergence with the ring of germs of holomorphic functions at $z = 0$. The elements of the latter are defined as equivalence classes of tuples $(U, f)$ consisting of an open set $U$ containing 0 and a holomorphic function $f\colon U \to \mathbb{C}$. Two such tuples $(U, f)$ and $(V, g)$ are equivalent whenever $f$ and $g$ agree on some subset of $U \cap V$ containing 0. Denote this ring by $R$. If a non-zero element in $R$ represented by $(U, f)$ satisfies $f(0) \neq 0$, then this element is a unit. On the other hand, if $f(0) = 0$, then there exist an integer $k \geqslant 1$ and unit $(V, h)$ such that $f = z^k h$. From this follows that every non-zero ideal is of the form $(z^k)$ for some $k$. Moreover $(z^k) \subset (z^\ell)$ precisely when $\ell \leqslant k$. Since every decreasing sequence of positive integers becomes constant, we see that $R$ is Noetherian.

(c) It is not Noetherian: Let $R$ denote the ring in question. This can be identified with the ring of holomorphic functions on $\mathbb{C}$. For each $n \in \mathbb{Z}_{>0}$ define

$$I_n = \{f \in R \mid f(k) = 0 \text{ for all } k \in \mathbb{Z} \text{ with } |k| \geqslant n\}$$

Then each $I_n$ is an ideal, and properly contained in $I_{n+1}$ (take for example (the continuation of) $\frac{\sin(\pi z)}{\prod_{k=-n}^{n}(z-k)}$). Thus $I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots$ is a strictly increasing sequence of ideals. Hence $R$ is not Noetherian.

(d) It is Noetherian: Let $S = \mathbb{C} + z^{k+1}\mathbb{C}[z] \subset \mathbb{C}[z]$ be the ring of polynomials whose first $k$ derivatives vanish at the origin. Consider the subring $R = \mathbb{C}[z^{k+1}]$. Then $S$ is generated as an $R$-module by the elements $1, z^{k+2}, \dots, z^{2k+1}$. Hence $S$ is Noetherian as an $R$-module (a quotient of a free $R$-module). But if a ring is Noetherian over a subring then it is already Noetherian.

(e) It is not Noetherian: Let $R = \mathbb{C}[z, zw, zw^2, zw^3, \dots] \subset \mathbb{C}[z, w]$. Then $R$ is the ring of polynomials $p$ such that $\frac{\partial p}{\partial w}(0, w) = 0$. Let $I_n := (z, zw, \dots, zw^n)$. We claim that $zw^{n+1} \notin I_n$, which implies that the $I_n$ form a strictly increasing sequence of ideals. Consider a general element $f = \sum_{k=0}^{n} b_k zw^k$ in $I_n$, where $b_k \in R$. Any monomial of $b_k$ which is divided by $w$ is also divided by $z$. Thus we find that all the monomials of $f$ are either a scalar multiple of $z^m w^n$ for some $m \geqslant 2$ or of $zw^k$ for some $0 \leqslant k \leqslant n$. Hence indeed $zw^{n+1} \notin I_n$.

4. Let $k$ be a field which is finitely generated as $\mathbb{Z}$-module. Prove that $k$ is a finite field.

*Solution*: Assume by contradiction that $\operatorname{char}(k) = 0$. Since $k$ is finitely generated as $\mathbb{Z}$-module and $\mathbb{Z}$ is Noetherian, we know that $k$ is Noetherian as module over $\mathbb{Z}$. But $\mathbb{Q} \subset k$ is a submodule and thus $\mathbb{Q}$ is finitely generated as $\mathbb{Z}$-module which we know is not true. A contradiction. Now assume that $\operatorname{char}(k) = p > 0$. Then $\mathbb{F}_p \subset k$. Since $k$ is finitely generated as $\mathbb{Z}$-module, it is finitely generated as $\mathbb{F}_p$-module. Thus $k$ is finite.

5. Let $X$ be a topological space. We say that $X$ is *Noetherian*, if the open subsets of $X$ satisfy the ascending chain condition, i.e. for every chain $U_i \subset U_{i+1}$ for $i \geqslant 0$ of open subsets, there is an integer $n \geqslant 0$ such that $U_i = U_{i+1}$ for all $i \geqslant n$. Prove:

   (a) A Noetherian space is quasi-compact.

   (b) If $A$ is a Noetherian ring, then $\operatorname{spec}(A)$ is Noetherian. Is the converse also true?

   *Solution*:

   (a) Let $\bigcup_{i\in I} V_i = X$ be an open covering of $X$. Without loss of generality we can assume $I = \mathbb{N}$. For $n \in \mathbb{N}$ define

   $$U_n := \bigcup_{i \leqslant n} V_i.$$

   We see that $U_n \subset U_{n+1}$ for all $n \in \mathbb{N}$. Since $X$ is Noetherian, we conclude that there is an integer $N \geqslant 0$ such that $U_i = U_{i+1}$ for all $i \geqslant N$, which implies that $U_N = \bigcup_{i\leqslant N} V_i = \bigcup_{i\in\mathbb{N}} V_i = X$. Hence there is a finite subcover and this proves that $X$ is quasi-compact.

   (b) We will use exercise 6 from exercise sheet 1. Let $(U_i)_{i\in\mathbb{N}}$ be an ascending chain of open subsets of $\operatorname{spec}(R)$. For every $i \in \mathbb{N}$ let $I_i \subset R$ be an ideal such that $\operatorname{spec}(R) \smallsetminus V(I_i) = U_i$. We conclude that $V(I_i) \supset V(I_{i+1})$ for all $i \in \mathbb{N}$. Therefore $I(V(I_i)) \subset I(V(I_{i+1}))$ for all $i \in \mathbb{N}$. Since $R$ is Noetherian we conclude that the chain of ideals $I(V(I_i))$ becomes stationary and thus the chain $V(I(V(I_i))) = V(I_i)$ becomes stationary. This implies that the chain $U_i$ becomes stationary.

   The converse is not true. Consider the ring $A := k[X_1, X_2, \dots]/(X_1^2, X_2^2, \dots)$ for a field $k$, i.e. the quotient of the polynomial ring in infinitely many variables modulo the ideal generated by all squares of variables. Let $\mathfrak{p} \subset A$ be prime ideal. Then the ideal $(X_1, X_2, \dots)$ is contained in $\mathfrak{p}$. But the ideal $(X_1, X_2, \dots)$ is already maximal. Hence $\operatorname{spec}(A)$ is only one point and trivially Noetherian as topological space. On the other hand, the ideals $I_n := (X_1, \dots, X_n)$ for $n \in \mathbb{N}$ form a strictly ascending chain in $A$, which proves that $A$ is not Noetherian.