Solutions Sheet 6

Primary Decomposition & k-Algebras

Definition: Let A be a ring and $\mathfrak{a} \subset A$ an ideal which admits a primary decomposition. Let P be the set of associated prime ideals of $\mathfrak{a}$. The minimal elements of P with respect to inclusion are called *isolated prime ideals* of $\mathfrak{a}$, the others are called *embedded prime ideals*.

1. The power of a maximal ideal is a primary ideal. Show that the converse is not true: find an example of a ring A and a primary ideal $\mathfrak{a} \subset A$ such that its radical $r(\mathfrak{a})$ is a maximal ideal, but $\mathfrak{a}$ is not a power of $r(\mathfrak{a})$.

   Solution: Consider the ring $A := \mathbb{Z}[X]$. We have the maximal ideal $\mathfrak{m} := (2, X)$ and the ideal $\mathfrak{a} := (4, X)$. Since $r(\mathfrak{a}) = \mathfrak{m}$ we know that $\mathfrak{a}$ is $\mathfrak{m}$-primary. However $\mathfrak{a} \not\subset \mathfrak{m}^2 = (4, X^2, 2X)$, so $\mathfrak{a}$ cannot be a power of $\mathfrak{m}$.

2. Let k be a field. Consider the polynomial ring $A := k[X, Y, Z]$ and the ideal $\mathfrak{a} := (X^2, XY, YZ, XZ) \subset A$. Find a minimal primary decomposition of $\mathfrak{a}$ and the associated prime ideals. Which components are isolated and which are embedded?

   Solution: We claim that $\mathfrak{a} = (X, Y) \cap (X, Z) \cap (X^2, Y^2, Z^2, XY, XZ, YZ) =: \mathfrak{b}$. Clearly $\mathfrak{a} \subset \mathfrak{b}$. On the other hand, let $f \in \mathfrak{b}$. Then we can write

   $$f = a_1X^2 + a_2Y^2 + a_3Z^2 + a_4XY + a_5XZ + a_6YZ$$

   for some $a_1, \ldots, a_6 \in A$. We see that $f \in \mathfrak{a}$ if and only if $a_2Y^2 + a_3Z^2 \in \mathfrak{a}$. Since $f \in (X, Z)$ we conclude that $a_2Y^2 \in (X, Z)$, so $a_2Y^2 \in (XY, ZY) \subset \mathfrak{a}$. Similarly we know that $f \in (X, Y)$ and thus $a_3Z^2 \in (XZ, YZ) \subset \mathfrak{a}$. Therefore $f \in \mathfrak{a}$ and hence $\mathfrak{a} = \mathfrak{b}$. Note that (X, Y) and (X, Z) are prime ideals and thus in particular primary. Furthermore, the ideal $(X^2, Y^2, Z^2, XY, XZ, YZ)$ is equal to $(X, Y, Z)^2$ and as power of a maximal ideal it is also primary. Thus we have found a primary decomposition of $\mathfrak{a}$ and we see that it is a minimal one:

   $$X \in (X, Y) \cap (X, Z), \text{ but } X \notin \mathfrak{a}$$
   $$Y^2 \in (X, Y) \cap (X, Y, Z)^2, \text{ but } Y^2 \notin \mathfrak{a}$$
   $$Z^2 \in (X, Z) \cap (X, Y, Z)^2, \text{ but } Z^2 \notin \mathfrak{a}$$

   The associated prime ideals are $(X, Y), (X, Z), (X, Y, Z)$. The ideals $(X, Y), (X, Z)$ are isolated primes and (X, Y, Z) is embedded.

3. Let k be a field and A a finitely generated k-algebra. Prove the following statement: An ideal $\mathfrak{a} \subset A$ is a maximal ideal if and only if $\mathfrak{a}$ is prime and the quotient $A/\mathfrak{a}$ is a finite dimensional k-vector space.

   Solution: Assume that $\mathfrak{a}$ is maximal. Then in particular it is a prime ideal. Furthermore, using the weak version of Hilbert's Nullstellensatz we conclude that $A/\mathfrak{m}$ is a finite field extension of k and thus finite dimensional over k. Conversely assume that $\mathfrak{a}$ is prime and $A/\mathfrak{a}$ is a finite dimensional k-vector space. Thus $A/\mathfrak{a}$ is an integral domain. Take a non-zero element $f \in A/\mathfrak{a}$. The multiplication by f gives a k-linear map

$$\begin{aligned} A/\mathfrak{a} &\to A/\mathfrak{a} \\ g &\mapsto fg \end{aligned}$$

   Since $A/\mathfrak{a}$ is an integral domain, this map is injective. Because of the finite dimensionality as k-vector space it is thus also surjective. We conclude that there is an element $g \in A/\mathfrak{a}$ such that $fg = 1$, hence $A/\mathfrak{a}$ is a field which proves that $\mathfrak{a}$ is a maximal ideal.

4. Let k be a field and A a finitely generated k-algebra. Let $\mathfrak{a} \subset A$ be a radical ideal. Using the previous exercise, prove that the associated prime ideals of $\mathfrak{a}$ are all maximal if and only if $A/\mathfrak{a}$ is a finite dimensional k-vector space.

   Solution: Since k is a field and A a finitely generated k-algebra we know that A is Noetherian, so primary decomposition exists.

   Assume that $A/\mathfrak{a}$ is finite dimensional as k-vector space. Using the previous exercise we conclude that every prime ideal in $A/\mathfrak{a}$ is maximal. Using the 1-to-1 correspondence of ideals in $A/\mathfrak{a}$ and ideals in A containing $\mathfrak{a}$ we conclude that every associated prime ideal of $\mathfrak{a}$ is maximal.

   Conversely assume that every associated prime ideal of $\mathfrak{a}$ is maximal. Since $\mathfrak{a}$ is radical we have $r(\mathfrak{a}) = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$ for distinct maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset A$. For all $1 \leqslant k \leqslant n$ consider the short exact sequence of k-vector spaces:

$$\mathfrak{m}_k/(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k) \hookrightarrow A/(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k) \twoheadrightarrow A/\mathfrak{m}_k$$

   Using the isomorphism theorem from algebra, we deduce that

$$\mathfrak{m}_k/(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_k) \cong (\mathfrak{m}_k + \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_{k-1})/(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_{k-1}) = A/(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_{k-1})$$

   where the last equality follows from the fact that the maximal ideals are distinct and the proposition from the lecture stating that if a prime ideal contains an intersection of ideals, it contains at least one of the ideals. We know from the previous exercise that $A/\mathfrak{m}_1$ is a finite dimensional k-vector space. Using induction, the above short exact sequence and the fact that if the left hand side and the right hand side of the short exact sequence are both finite dimensional, then so is the middle term, we deduce that $A/\mathfrak{a}$ is a finite dimensional k-vector space.

5. For $k := \mathbb{C}$, explain the geometry behind exercise 2.

   Solution: We look at $V(\mathfrak{a}) \subset \mathbb{C}^3$. By definition, this is the set of $x \in \mathbb{C}^3$ such that $\forall f \in \mathfrak{a}$ we have $f(x) = 0$. It is enough to check that condition on the generators. Hence $x \in V(\mathfrak{a})$ if and only if $X = 0$ and $YZ = 0$. We conclude that $V(\mathfrak{a})$ is in fact the union of the Y-axis (when $X, Z = 0$) and the Z-axis (when $X, Y = 0$). Those are two irreducible components of $V(\mathfrak{a})$. The Y-axis corresponds to the prime ideal (X, Z) and the Z-axis corresponds to the ideal (X, Y). Thus we recovered the isolated prime ideals of $\mathfrak{a}$.