D-MATH
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Commutative Algebra

HS 2017

Solutions Sheet 7

Primary Decomposition, Artinian Rings and Modules

Definition: Let A be a ring and $\mathfrak{p} \subset A$ a prime ideal of A. Denote $\varphi : A \to A_{\mathfrak{p}}$ for the localisation map. For an integer $n > 0$ we define the *n-th symbolic power of* $\mathfrak{p}$ to be the ideal $\mathfrak{p}^{(n)} := \varphi^* \varphi_*(\mathfrak{p}^n)$.

1. Let A be a ring and $\mathfrak{p} \subset A$ a prime ideal. Let $n > 0$ be an integer. Consider the n-th symbolic power $\mathfrak{p}^{(n)}$. Prove:

   (a) $\mathfrak{p}^{(n)}$ is a $\mathfrak{p}$-primary ideal.

   (b) If $\mathfrak{p}^n$ has a primary decomposition, then $\mathfrak{p}^{(n)}$ is its $\mathfrak{p}$-primary component.

   (c) $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ if and only if $\mathfrak{p}^n$ is a primary ideal.

   Solution:

   (a) Since $\varphi_*(\mathfrak{p})$ is a maximal ideal in $A_{\mathfrak{p}}$ we know that $\varphi_*(\mathfrak{p})^n$ is a $\varphi_*(\mathfrak{p})$-primary ideal. But $\varphi_*(\mathfrak{p})^n = \varphi_*(\mathfrak{p}^n)$ (see solutions of exercise 1, sheet 2) and since the contraction of a primary ideal is primary, we conclude that $\mathfrak{p}^{(n)}$ is a primary ideal.

   (b) Let $\mathfrak{p}^n = \bigcap_{i=1}^n \mathfrak{q}_i$ be a minimal primary decomposition and set $\mathfrak{p}_i := r(\mathfrak{q}_i)$ for all $1 \leqslant i \leqslant n$. Since $r(\mathfrak{p}^n) = \mathfrak{p}$ we conclude that $\mathfrak{p}$ is a minimal prime ideal associated to $\mathfrak{p}^n$, and it is in fact the only one. Without loss of generality assume that $\mathfrak{q}_1$ is the $\mathfrak{p}$-primary component of the decomposition. Since $\mathfrak{p}^n \subset \mathfrak{p}^{(n)}$ we see that we can exchange $\mathfrak{q}_1$ in the decomposition by $\mathfrak{q}_1 \cap \mathfrak{p}^{(n)}$, which is still a $\mathfrak{p}$-primary ideal by a Lemma from the lecture. By the second uniqueness theorem we thus conclude that $\mathfrak{q}_1 = \mathfrak{q}_1 \cap \mathfrak{p}^{(n)}$ and so $\mathfrak{q}_1 \subset \mathfrak{p}^{(n)}$. On the other hand $\mathfrak{p}^n \subset \mathfrak{q}_1$ and so $\mathfrak{p}^{(n)} \subset \varphi^*\varphi_*(\mathfrak{q}_1)$. But by a Proposition we have the equality $\varphi^*\varphi_*(\mathfrak{q}_1) = \mathfrak{q}_1$. This concludes the proof.

   (c) If $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, then by (a) the ideal $\mathfrak{p}^n$ is primary. Conversely, if $\mathfrak{p}^n$ is primary, then it has itself as trivial primary decomposition. Using (b) we conclude that $\mathfrak{p}^{(n)} = \mathfrak{p}^n$.

2. Let k be a field and consider the ring $A := k[X, Y, Z]$. Compute the ideal of A given by

$$\mathfrak{a} := (Y^2 + XY - XZ - YZ, (X+Y)^2 + 2X) \cap ((X+Y)^2, X, Y^3 - Y^2 Z)$$

   and find a minimal primary decomposition of $\mathfrak{a}$.

   (*Hint:* Substitutions might be helpful.)

Solution: We substitute $U := X + Y$ and $V := Y - Z$ and use the fact that we can change one generator by a multiple of another one, thus finding:

$$\begin{aligned}\mathfrak{a} &= (UV, U^2 + 2X) \cap (U^2, X, Y^2V) \\ &= (UV, U^2 + 2X) \cap (U^2, X, U^2V - X^2V - 2XYV) \\ &= (UV, U^2 + 2X) \cap (U^2, X)\end{aligned}$$

Substituting $W := U^2 + 2X$ we get:

$$\mathfrak{a} = (UV, W) \cap (U^2, W)$$

Now we see that $\mathfrak{a} = (U^2V, W)$. We have the decomposition

$$\mathfrak{a} = (U^2, W) \cap (V, W)$$

which is in fact a minimal primary decomposition, as both (U^2, W) and (V, W) are primary ideals with different radicals. Substituting back and doing a bit of cosmetics, we get the primary decomposition:

$$\mathfrak{a} = (Y^2, X) \cap (Y - Z, (X + Y)^2 + 2X).$$

3. Let A be a ring and $\mathfrak{a} \subset A$ an ideal which admits a primary decomposition. Let $\mathfrak{p}$ be a maximal element of the set $\{(\mathfrak{a} : x) \mid x \in A \smallsetminus \mathfrak{a}\}$. Prove that $\mathfrak{p}$ is an associated prime ideal of $\mathfrak{a}$.

   Solution: By the first uniqueness theorem, we know that every prime ideal of the form $(\mathfrak{a} : x)$ for some $x \in A$ is an associated prime ideal. Thus we only need to proof that $\mathfrak{p}$ is indeed a prime ideal. Let $x \in A \smallsetminus \mathfrak{a}$ such that $\mathfrak{p} = (\mathfrak{a} : x)$. Let $f, g \in A$ such that $fg \in \mathfrak{p}$. Then $fgx \in \mathfrak{a}$ and we see that $\mathfrak{p} \subset (\mathfrak{a} : gx)$. If $\mathfrak{p} = (\mathfrak{a} : gx)$, then $f \in \mathfrak{p}$. If not, then by maximality of $\mathfrak{p}$ we conclude that $gx \in \mathfrak{a}$. But this implies that $g \in \mathfrak{p}$. Hence $\mathfrak{p}$ is a prime ideal.

4. Let A be a ring and M an Artinian A-module. Let $f : M \to M$ be an injective module homomorphism. Prove that f is an isomorphism.

   Solution: Consider the descending chain of submodules $I_n := \operatorname{im}(f^n)$ for $n > 0$. Since M is Artinian, we conclude that there exists an integer $N > 0$ such that for all $n \geqslant N$ we have $I_n = I_{n+1}$. Let $a \in M$. We conclude that there is an element $b \in M$ such that $f^{n+1}(b) = f^n(a)$. Hence we have $f^n(f(b) - a) = 0$. Since f is injective, so is f^n and thus $f(b) = a$, proving that f is surjective. We conclude that f is an isomorphism.

5. Let A be a ring and M a Noetherian A-module. Let $\mathfrak{a} \subset A$ be the annihilator of M, i.e. $\mathfrak{a} = \{x \in A \mid xM = 0\}$. Prove that the ring $A/\mathfrak{a}$ is Noetherian.

   Is the same true if we replaced Noetherian with Artinian?

Solution: Since M is Noetherian, it is finitely generated by, say, the elements $m_1 \dots, m_n$. Consider the homomorphism

$$\begin{aligned} \varphi : A &\to M^n \\ a &\mapsto (am_1, \dots, am_n) \end{aligned}$$

The kernel of φ is precisely the annihilator $\mathfrak{a}$. Thus $A/\mathfrak{a}$ can be identified with a submodule of M^n. But since M is Noetherian, so is M^n and so is every submodule of M^n. We conclude that $A/\mathfrak{a}$ is Noetherian as an A-module. Hence it is Noetherian as a ring.

If we replaced Noetherian with Artinian, the statement becomes wrong. Consider $M := \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ as a $\mathbb{Z}$-module by multiplication. Clearly the annihilator $\mathfrak{a}$ is zero, as no integer is divisible by every power of p. On the other hand M is Artinian: Let $\frac{a}{p^n} \in M$ with a and p^n coprime. By the Lemma of Bézout we conclude that there are $u, v \in \mathbb{Z}$ such that $ua - vp^n = 1$. But then $\frac{ua}{p^n} = \frac{1}{p^n} + v$, which shows that $\frac{ua}{p^n} \equiv \frac{1}{p^n}$ in M. Hence every descending chain of submodules is of the form $\left\langle \frac{1}{p^n} \right\rangle \supset \left\langle \frac{1}{p^k} \right\rangle \supset \dots$ with $k \leqslant n$. We conclude that M is Artinian. This provides a counter example, as $\mathbb{Z} = \mathbb{Z}/\mathfrak{a}$ is not Artinian.