Solutions Sheet 8

Artinian Rings and Modules, Krull Intersection Theorem

1. Let k be a field and A a finitely generated k-algebra. Prove that the following are equivalent:

   (a) A is an Artinian ring.

   (b) A is a finite dimensional k-vector space.

   Solution: "(a)⇒(b)": If A is Artinian, then it is isomorphic to a product $A \cong \prod_{i=1}^{n} A_i$ of local Artinian rings A_i by the classification theorem. So without loss of generality we assume that A is a local Artinian ring with maximal ideal $\mathfrak{m} \subset A$. By Hilbert's Nullstellensatz we know that $L := A/\mathfrak{m}$ is a finite field extension of k. From one Proposition we know that $\mathfrak{m}^n = 0$ for some integer $n > 0$. For every $1 \leqslant i \leqslant n$ we have $L \otimes_A \mathfrak{m}^i \cong \mathfrak{m}^i/\mathfrak{m}^{i+1}$. Since A is also Noetherian, $\mathfrak{m}^i$ is finitely generated as A-module and thus $L \otimes_A \mathfrak{m}$ is finitely generated as L-vector space and hence also as k-vector space. For $1 \leqslant i \leqslant n$ we have the exact sequences

   $$\mathfrak{m}^i/\mathfrak{m}^{i+1} \to A/\mathfrak{m}^{i+1} \to A/\mathfrak{m}^i$$

   where the left term is a finite dimensional k-vector space. For $i = 1$, the right hand side is L and thus also finite dimensional as k-vector space, so the middle term $A/\mathfrak{m}^2$ is finite dimensional as k-vector space. By induction and the fact that $\mathfrak{m}^n = 0$ we conclude that $A/\mathfrak{m}^n = A$ is finite dimension as k-vector space.

   "(b)⇒(a)": If A is a finite dimensional k-vector space, then it is Artinian as k-vector space. But every ideal of A is a k-vector space and thus they satisfy the descending chain condition, which proves that A is Artinian as ring.

2. Let A be a Noetherian ring. Prove the equivalence of the following statements:

   (a) A is an Artinian ring.

   (b) $\operatorname{spec}(A)$ is discrete and finite.

   (c) $\operatorname{spec}(A)$ is discrete.

   Solution: "(a)⇒(b)": If A is Artinian, we know that every prime ideal is maximal. Hence every point of $\operatorname{spec}(A)$ is closed. We also know, that there are only finitely many distinct maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of A. Hence $\operatorname{spec}(A)$ is finite. Furthermore $\operatorname{spec}(A) \setminus V(\bigcap_{i \neq k} \mathfrak{m}_i) = \mathfrak{m}_k$ and thus every point is also closed. We conclude that $\operatorname{spec}(A)$ is a discrete finite space.

"(b)⇒(c)": Immediate.

"(c)⇒(a)": We show that $\dim(A) = 0$. Assume otherwise. Then there are two prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \subset A$ such that $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq A$. But then $\mathfrak{p}_1 \in \operatorname{spec}(A)$ is not a closed point, since its closure contains $\mathfrak{p}_2$. A contradiction to $\operatorname{spec}(A)$ being discrete. Since A is Noetherian of dimension 0 we conclude that A is Artinian.

3. Let R be the ring of germs at 0 of C^∞-functions on $\mathbb{R}$ and let $\mathfrak{m} = (x)$ denote the ideal generated by the coordinate function x. It is the unique maximal ideal of R. Show by elementary calculus that if f is a C^∞-function such that all its derivatives vanish at the origin, then f/x is also such a function. Conclude that $\mathfrak{m}\left(\bigcap_{j\geqslant 1}\mathfrak{m}^j\right) = \bigcap_{j\geqslant 1}\mathfrak{m}^j$. Moreover, recall that the intersection $\bigcap_{j\geqslant 1}\mathfrak{m}^j$ is non-zero. It contains for instance the function $e^{-\frac{1}{x^2}}$. This example therefore shows that the finiteness conditions are necessary in both Nakayama's Lemma and Krull's Intersection Theorem.

   Solution (sketch): We only sketch the proof, which is purely analytic. Firstly, one can show by induction on k and using Bernoulli de l'Hopital that for any such f and any $k \geqslant 0$ the first derivative of f/x^k at 0 exists and equals 0. Secondly, one can show that the same is true for any j-th derivative, where $j \geqslant 1$, which can be done by induction on j using the first part. This implies $f/x \in \mathfrak{m}^j$ for any $j \geqslant 1$ and consequently $f \in \mathfrak{m}(\bigcap\limits_{j\geqslant 1}\mathfrak{m}^j)$. As all of the derivatives at 0 of any $f \in \bigcap\limits_{j\geqslant 1}\mathfrak{m}^j$ vanish, we deduce $\bigcap\limits_{j\geqslant 1}\mathfrak{m}^j \subset \mathfrak{m}(\bigcap\limits_{j\geqslant 1}\mathfrak{m}^j)$.

4. Consider the monoid ring $R := K[\mathbb{Q}^{\geqslant 0}] = K[\{X^\alpha \mid \alpha \in \mathbb{Q}^{\geqslant 0}\}]$ over a field K.

   (a) Find a maximal ideal $\mathfrak{m} \neq 0$ of R such that $\mathfrak{m}^n = \mathfrak{m}$ for all $n \geqslant 1$.

   (b) Use R to construct a ring $\bar{R}$ with $r(0)^n \neq 0$ for all $n \geqslant 1$.

   Solution: (a) Let $\mathfrak{m}$ be the ideal generated by the elements X^α for all $\alpha \in \mathbb{Q}^{>0}$. It is maximal, because the factor ring is isomorphic to K. Since $X^\alpha = (X^{\alpha/n})^n \in \mathfrak{m}^n$ for each $n \geqslant 1$, we have $\mathfrak{m}^n = \mathfrak{m}$.

   (c) For any rational number $\alpha > 0$ with denominator n we have $(X^\alpha)^n \in (X)$. Thus for $\bar{R} := R/(X)$ we have $r(0) = \mathfrak{m}/(X)$. From (a) it follows that $r(0)^n = r(0) \neq 0$ for all $n \geqslant 1$.