D-MATH | Commutative Algebra | HS 2017
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Solutions Sheet 9

Dimension and Height

1. Let A be a ring. Prove the following statements:

   (a) For every prime ideal $\mathfrak{p} \subset A$ we have $\mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) \leqslant \dim(A)$.

   (b) For every ideal $\mathfrak{a} \subset A$ we have $\mathrm{ht}(\mathfrak{a}) + \mathrm{coht}(\mathfrak{a}) \leqslant \dim(A)$.
   (Recall that $\mathrm{ht}(\mathfrak{a}) = \inf\limits_{\mathfrak{p} \supset \mathfrak{a} \text{ prime}} \mathrm{ht}(\mathfrak{p})$)

   Solution:

   (a) Let $\varphi : A \to A_{\mathfrak{p}}$ and $\psi : A \to A/\mathfrak{p}$ be the canonical homomorphisms. By definition we have $\mathrm{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}})$ and $\mathrm{coht}(\mathfrak{p}) = \dim(A/\mathfrak{p})$. For all chains of prime ideals $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$ in $A_{\mathfrak{p}}$ and $\mathfrak{p} = \mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_r$ in $A/\mathfrak{p}$ we have the chain $\varphi^*(\mathfrak{p}_0) \subsetneq \cdots \subsetneq \varphi^*(\mathfrak{p}_n) = \psi^*(\mathfrak{q}_0) \subsetneq \cdots \subsetneq \psi^*(\mathfrak{q}_r)$ of prime ideals of A. The length of this chain is thus $n + r \leqslant \dim(A)$. Taking the supremum over all such pairs of chains yields the result.

   (b) Let $\mathfrak{p} \subset A$ be a prime ideal. Then by definition of $\mathrm{ht}(\mathfrak{a})$ and (a) above we have
   $$\mathrm{ht}(\mathfrak{a}) + \mathrm{coht}(\mathfrak{p}) \leqslant \mathrm{ht}(\mathfrak{p}) + \mathrm{coht}(\mathfrak{p}) \leqslant \dim(A)$$
   Taking the supremum of the left hand side over all prime ideals $\mathfrak{p}$ which contain $\mathfrak{a}$ yields the result:
   $$\mathrm{ht}(\mathfrak{a}) + \mathrm{coht}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{a}) + \sup_{\mathfrak{p} \supset \mathfrak{a} \text{ prime}} \mathrm{coht}(\mathfrak{p}) \leqslant \dim(A)$$

2. Give an example of a

   (a) non-Noetherian local ring A with maximal ideal $\mathfrak{m}$ and $\dim A > \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

   (b) Noetherian non-local ring A with a maximal ideal $\mathfrak{m}$ such that $\dim A > \mathrm{ht}(\mathfrak{m})$.

   Solution: Let k be a field.

   (a) The localisation A of the ring $k[X^\alpha|_{\alpha \in \mathbb{Q}^{>0}}]$ at the prime ideal $(X^\alpha|_{\alpha \in \mathbb{Q}^{>0}})$ satisfies $\dim(A) = 1 > 0 = \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

   (b) Since the dimension of a ring is the supremum of the heights of maximal ideals it is enough to find a ring with two maximal ideals of different heights. Exercise 4 gives an example.

3. Let A be a ring and consider the polynomial ring in one variable $A[X]$. Let $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subset A[X]$ be two prime ideals such that their contraction to A is equal $\mathfrak{p} := \mathfrak{p}_1 \cap A = \mathfrak{p}_2 \cap A$. Prove that $\mathfrak{p}_1 = \mathfrak{p}A[X]$. Deduce that for any three subsequent prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3 \subset A[X]$ their contractions $\mathfrak{p}_1 \cap A, \mathfrak{p}_2 \cap A, \mathfrak{p}_3 \cap A$ cannot all be equal.

   [Hint: Take a ring of fractions and use that $\dim(K[X]) = 1$ for every field K]

   Solution: Since $\mathfrak{p}_1$ is an ideal we have $\mathfrak{p}A[X] \subset \mathfrak{p}_1$. We localise $A[X]$ at the multiplicative set $S := A \setminus \mathfrak{p}$ and get $S^{-1}A[X] = A_{\mathfrak{p}}[X]$ and the two prime ideals $S^{-1}\mathfrak{p}_1 \subsetneq S^{-1}\mathfrak{p}_2$ with contractions $S^{-1}\mathfrak{p}_1 \cap A_{\mathfrak{p}} = S^{-1}\mathfrak{p}_2 \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. Now consider the quotient $R := A_{\mathfrak{p}}[X]/\mathfrak{p}A_{\mathfrak{p}}[X] \cong (A_{\mathfrak{p}}/p)[X]$. Since $A_{\mathfrak{p}}/\mathfrak{p}$ is a field we know that the dimension of R is equal to 1. We conclude that therefore the image of $S^{-1}\mathfrak{p}_1$ in R is the zero ideal (because it cannot be equal to the image of $S^{-1}\mathfrak{p}_2$) and thus $S^{-1}\mathfrak{p}_1 = \mathfrak{p}A_{\mathfrak{p}}[X]$. By the correspondence of ring of fractions we conclude that $\mathfrak{p}_1 = \mathfrak{p}A[X]$.

   Assume we have three subsequent prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3 \subset A[X]$ such that all of their contractions to A are equal to the prime ideal $\mathfrak{p} \subset A$. By using the previous proof twice we conclude that $\mathfrak{p}_2 = \mathfrak{p}A[X] = \mathfrak{p}_1$ which is a contradiction.

4. Consider the ring $R := \mathbb{C}[X, Y, Z]/(XY, XZ)$. Compute the height of the two maximal ideals $\mathfrak{m}_1 := (X - 1, Y, Z) \subset R$ and $\mathfrak{m}_2 := (X, Y - 1, Z) \subset R$. Interpret your result geometrically on the variety $V(XY, XZ) \subset \mathbb{C}^3$.

   Solution: Note that the ideals are maximal by the correspondence of ideals in R and ideals in $\mathbb{C}[X, Y, Z]$ containing (XY, XZ). Furthermore note that $(XY, XZ) = (X) \cap (Y, Z)$ in the ring $\mathbb{C}[X, Y, Z]$ and thus every prime ideal of R corresponds to a prime ideal in $\mathbb{C}[X, Y, Z]$ containing (X) or (Y, Z).

   We have the chain of prime ideals $\mathfrak{m}_1 = (X - 1, Y, Z) \supsetneq (Y, Z)$. Every prime ideal $\mathfrak{p} \subset \mathfrak{m}_1$ must contain (Y, Z) because $X \notin \mathfrak{m}_1$. Furthermore, there is no prime ideal in between $\mathfrak{m}_1$ and (Y, Z), since every $f \in \mathfrak{m}_1 \setminus (Y, Z)$ must be divisible by $(X - 1)$, say $f = a(X - 1)$ for some $a \in R \setminus (Y, Z)$. If $(X - 1)$ is contained in the ideal, then we are done, if not, then $a \in \mathfrak{m}_1$ and we can use induction on the degree. Hence the chain is maximal and we conclude that $\mathrm{ht}(\mathfrak{m}_1) = 1$.

   Consider the chain of prime ideals $\mathfrak{m}_2 = (X, Y - 1, Z) \supsetneq (X, Z) \supsetneq (X)$. We conclude that $\mathrm{ht}(\mathfrak{m}_2) \geqslant 2$. Since $Y \notin \mathfrak{m}_2$ we conclude that every prime ideal of R contained in $\mathfrak{m}_2$ corresponds to a prime ideal in $\mathbb{C}[X, Y, Z]$ which contains X. Thus $\mathrm{ht}(\mathfrak{m}_2) \leqslant \dim(R/(X)) = \dim(\mathbb{C}[Y, Z]) = 2$. Hence $\mathrm{ht}(\mathfrak{m}_2) = 2$.

   For the geometric interpretation note that $V(XZ, YZ)$ is the union of the X-axis and the YZ-plane in $\mathbb{C}^3$. The point which corresponds to $\mathfrak{m}_1$ is a point on the X-axis and thus there is only one irreducible subspace which contains that point, namely the X-axis itself which corresponds to the ideal (Y, Z). On the other hand, the point which corresponds to $\mathfrak{m}_2$ lies on the YZ-plane and more precisely on the Y-axis. Thus there is the irreducible variety of the Y-axis (corresponding to the ideal (X, Z) and the irreducible variety of the YZ-plane (corresponding to the ideal (X)) which contains the Y.axis and thus the point. This also shows geometrically the results $\mathrm{ht}(\mathfrak{m}_1) = 1$ and $\mathrm{ht}(\mathfrak{m}_2) = 2$.