Haar measures

Student seminar on homogeneous dynamics

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Friday 20 October and Monday 27 October, 2017

Contents

DISCLAIMER: This document contains the mistakes it contains . . .

References

If not stated otherwise, in-text references are to Einsiedler–Ward.

- M. Einsiedler, T. Ward, *Ergodic Theory with a View towards Number Theory* (Springer-Verlag Londen Limited, 2011)
  - · §8.3 describes Existence of Haar measures (sketch)
  - · §8.3.1 describes "Uniqueness" of Haar measures
  - · §C.2 describes Modular function, examples of Haar measures
- G.B. Folland, *A course in abstract harmonic analysis*, in *Studies in Advanced Mathematics* (CRC Press, Boca Raton, FL, 1995)
  - · Section 2.2 describes details on the existence of Haar measures
- B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's property (T)* (Cambridge University Press, 2008)
  - · §A.3, especially interesting are Remark A.3.1 p. 300 and Proposition A.3.3 p. 302.

- Hewitt, Ross, *Abstract Harmonic Analysis Volume I, second edition* (Springer-Verlag, 1979)
    - · Chapter 15, in particular Theorem 15.5 and its proof, pp. 185–193.
- J-F. Quint, *Systèmes dynamiques dans les espaces homogènes* (Lecture notes, février 2012)
    - · §3.2 Mesure de Haar
- T. Tao, *245C, Notes 2: The Fourier transform* (https://terrytao.wordpress.com/2009/04/06/the-fourier-transform/#gelfand, viewed 16 October 2017)
    - · In particular Exercise 2
- Course: Introduction to Lie Groups

1 Recap left-invariant Haar measures

Theorem 1 (Haar)**.** *For all locally compact groups G, there exists a non-zero, left-invariant Radon measure m_G on the Borel σ-algebra $\mathcal{B}_G$. That means that it has the following properties:*

- $\forall A \in \mathcal{B}_G, \forall g \in G : m_G(gA) = m_G(A)$ *(left-invariance)*
- *It is finite on compact sets (locally finite)*
- $\forall A \in \mathcal{B}_G : m_G(A) = \sup \{m_G(K) : A \supset K, K \subset G \text{ compact}\}$ *(inner regularity)*
- $\forall A \in \mathcal{B}_G : m_G(A) = \inf \{m_G(B) : A \subset B, B \subset G \text{ open}\}$ *(outer regularity)*

We have seen that such measures are positive on non-empty open sets.

The following theorem characterises measures with the above properties:

Theorem 2 (Part of Theorem C.4 (Haar) p. 431, see also Theorem (Haar) p. 243)**.** *Let G be as in Theorem 1. Then left-invariant measures that are positive on non-empty open sets and bounded on compact sets are unique up to scalingby a $C \in \mathbb{R}_{>0}$.* [1]

"Uniqueness" will be proved below, for the existence we refer to Folland Section 2.2 (or one of the other references). [2]

[1] In addition, it can be shown that $m_G(G) < \infty$ if and only if G is compact (homework exercise). For compact G it is usual to use the normalisation $m_G(G) = 1$.

[2] For existence §8.3 in Einsiedler-Ward sketches the following argument, referring to Folland for details. Existence is based on a covering argument: fix a compact set K_0 and take an arbitrary open set V. Now one can calculate the minimal amount of translations of V needed to cover K resp. K_0. For carefully chosen, small open sets V it can be shown that the ratio of these two numbers behaves nicely. In particular, one can show that the limit of this ratio for arbitrarily small open sets V exists, which is then used to define a measure on the Borel σ-algebra and to show the desired properties. The scaling of this Haar measure depends on the choice of K_0, which by construction has measure 1.

2 Proof of uniqueness up to scaling

We will now prove the second part of Theorem 2. First we derive some results from the theorem of Fubini–Tonelli for non-negative, measurable functions. The second step uses the existence and uniqueness of Radon-Nikodym derivatives for absolutely continuous measures. [3]

With slight abuse of notation we write "Haar measure" for a measure satisfying the properties described in Theorem 2. Once we have proved the "uniqueness" we know that both notions are equivalent.

Proposition 3 (Part of Corollary 8.6 pp. 246–247)**.** *Let G be a σ-locally compact group with left-invariant Haar measure m_G. Then for any two Borel sets $B_1, B_2 \in \mathcal{B}_G$ with $m_G(B_1), m_G(B_2) > 0$ we define $\mathcal{O} := \{g \in G : m_G(gB_1 \cap B_2) > 0\}$ and find that $m_G(\mathcal{O}) > 0$.* [4] *Moreover,*

$$\forall B \in \mathcal{B}_G : m_G(B) > 0 \Leftrightarrow m_G(B^{-1}) > 0.$$

Proof. We note the following:

$$h \in gB_1 \Leftrightarrow g \in hB_1^{-1}$$

Together with the theorem of Fubini–Tonelli we find: [5]

$$\begin{aligned}
\int m_G(gB_1 \cap B_2)\, \mathrm{d}m_G(g) &= \iint \chi_{gB_1}(h)\chi_{B_2}(h)\, \mathrm{d}m_G(h)\, \mathrm{d}m_G(g) \\
&= \int \chi_{B_2}(h) \int \chi_{hB_1^{-1}}(g)\, \mathrm{d}m_G(g)\, \mathrm{d}m_G(h) \\
&= m_G(hB_1^{-1}) \int \chi_{B_2}(h)\, \mathrm{d}m_G(h) \\
&= m_G(B_1^{-1}) m_G(B_2)
\end{aligned}$$

It proves the last part of the proposition: with $B_2 = G$, $m_G(G) > 0$ and $0\infty = 0$ we find:

$$\begin{aligned}
& m_G(B_1)m_G(G) = m_G(B_1^{-1})m_G(G) \\
\Rightarrow \quad & \forall B_1 \in \mathcal{B}_G : m_G(B_1) > 0 \Leftrightarrow m_G(B_1^{-1}) > 0
\end{aligned}$$

Using that we get:

$$\int m_G(gB_1 \cap B_2)\, \mathrm{d}m_G(g) = m_G(B_1^{-1})m_G(B_2) > 0 \;\Rightarrow\; m_G(\mathcal{O}) > 0$$

□

[3] Here we only use σ-finiteness of G, and left-invariance of $m_G \not\equiv 0$. See Hewitt-Ross Section 15 for a proof of "uniqueness" that only uses local compactness of G. One could hope that Haar measures exist on a larger class of spaces than the ones we study. A result in this direction is due to Weil, see Bekka–de la Harpe–Valette Remark A.3.1.

[4] Below it is shown that $\mathcal{O}$ is open. In addition, the statement also holds for $\mathcal{O} = \{g \in G : m_G(B_1 g \cap B_2) > 0\}$.

[5] This is Fubini for non-negative, not necessarily integrable functions, a.k.a. the theorem of Fubini–Tonelli. Note that Theorem A.13 p. 409 is also called Fubini–Tonelli, but covers the case of integrable functions.

Proof of Theorem 2. (this is part of the proof of Corollary 8.8 pp. 248–249)

Let m_1, m_2 be two left-invariant Haar measures and let $m := m_1 + m_2$, which is then also a left-invariant Haar measure. In addition, m_1, m_2 are absolutely continuous with respect to m. [6]This implies that m_1, m_2 have (unique) densities $f_1, f_2 : G \to \mathbb{R}_{\geq 0}$ (measurable) such that $\mathrm{d}m_i = f_i \,\mathrm{d}m$ for $i \in \{1, 2\}$ (Theorem A.15). [7]

What rests to be shown is that f_1, f_2 are constant m-a.e. Proof by contradiction: assume f_1 is not. Then there exist $B_1, B_2 \in \mathcal{B}_G$ with $m(B_1), m(B_2) > 0$ and [8]

$$\forall x_1 \in B_1, x_2 \in B_2 : f_1(x_1) < f_1(x_2).$$

Proposition 3 for $m_G = m$ gives a $g \in G$ such that $0 < m(gB_1 \cap B_2) = m(B_1 \cap g^{-1}B_2)$. For $x \in B_1 \cap g^{-1}B_2$ we have $f_1(x) < f_1(gx)$.

On the other hand, we can use the left-invariance of m_1 and m to find (for g as before):

$$\forall E \in \mathcal{B}_G : \int_E f_1(x) \,\mathrm{d}m(x) = m_1(E) = m_1(g^{-1}E) = \int_{g^{-1}E} f_1 \,\mathrm{d}m = \int_E f_1(gx) \,\mathrm{d}m(x)$$

The fact that it holds for all E and the uniqueness of Radon-Nikodym derivatives gives $f_1(x) = f_1(gx)$ m-a.e. This contradicts the fact that $f_1(\cdot) < f_1(g\,\cdot)$ on a set of positive measure, and thus shows that f_1 is constant m-a.e. The same holds for f_2. Together we find $m_1 = \frac{f_1}{f_2} m_2$, where $f_1, f_2 \in \mathbb{R}_{>0}$. □

3 The modular function (§C.2)

Given a left-invariant Haar measure m_G and a $g \in G$, we can look at the following:

$$\forall A \in \mathcal{B}_G : \mu(A) := m_G(Ag)$$

Note that μ is a left-invariant Haar measure as well, by Theorem 2 we know that $\exists \Delta(g) \in \mathbb{R}_{>0} : \forall A \in \mathcal{B}_G : m_G(Ag) = \Delta(g) m_G(A)$. This defines the *modular function*:

$$\Delta : G \to \mathbb{R}_{>0}$$

Theorem 2 implies that this definition is independent of the choice of left-invariant Haar measure m_G. It can be shown that Δ is continuous, here we just highlight that it is a homomorphism:

$$\Delta(hg) = \frac{m_G(Ahg)}{m_G(A)} = \frac{m_G(Ahg)}{m_G(Ah)} \cdot \frac{m_G(Ah)}{m_G(A)} =: \frac{m_G(Bg)}{m_G(B)} \cdot \frac{m_G(Ah)}{m_G(A)} = \Delta(g)\Delta(h)$$

for any $A \in \mathcal{B}_G$ with $0 < m_G(A) < \infty$ and $B := Ah$.

G is called *unimodular* if $\Delta \equiv 1$, which is equivalent to left-invariant Haar measures also being right-invariant.

[6]Absolutely continuous means that all m-null sets are also m_1, m_2-null.

[7]These are called Radon-Nikodym derivatives of m_i with respect to m.

[8]To see this, look at the pre-images $f_1^{-1}([\frac{k}{n}, \frac{k+1}{n}))$, the fact that f_1 is not constant implies that for suitable n two of them have non-zero measure.

Examples of unimodular groups:

- Abelian groups
- Discrete groups [9]
- Compact groups [10]
- Semi-simple Lie groups [11]

4 Specific examples of Haar measures

Example 4 (Partly corresponds to Example C.5)**.**

- *The Lebesgue measure* λ *on* $(\mathbb{R}^n, +)$. [12]
- *The same is true for the torus* $(\mathbb{T}^n, +)$.
- *For discrete groups (groups with the discrete topology) the counting measure is a Haar measure.* [13]

Example 5 (Partly corresponds to Example C.5)**.**

- $GL_d(\mathbb{R}) \subset \mathbb{R}^{d^2}$ *with the relative topology. We identify an element* $x \in GL_d(\mathbb{R})$ *with the vector containing its components* $(x_{11}, x_{12}, \ldots, x_{dd}) \in \mathbb{R}^{d^2}$. *Up to scaling Haar measures on* $GL_d(\mathbb{R})$ *are defined by:*

$$\mathrm{d}m_G(x) := \frac{\mathrm{d}x_{11}\,\mathrm{d}x_{12}\cdots\,\mathrm{d}x_{dd}}{|\det x|^d}$$

We try the following Ansatz:

$$\begin{aligned}
\frac{\mathrm{d}x_{11}\,\mathrm{d}x_{12}\cdots\,\mathrm{d}x_{dd}}{F(x)} &= \mathrm{d}m_G(x) \\
&= \mathrm{d}m_G(gx) \\
&= \frac{\mathrm{d}(gx)_{11}\,\mathrm{d}(gx)_{12}\cdots\,\mathrm{d}(gx)_{dd}}{F(gx)} \\
&= \frac{\mathrm{d}x_{11}\,\mathrm{d}x_{12}\cdots\,\mathrm{d}x_{dd}}{F(gx)}\,|\det \mathrm{Jac}\,|
\end{aligned}$$

It rests to find the function F. *The Jacobian of the coordinate change* $y = g \bullet x$ *is*

[9] On which the counting measure is a Haar measure.

[10] The only compact subgroup of $\mathbb{R} > 0$ is $\{1\}$, so the image of the compact group G under the continuous map Δ is $\{1\}$, which shows that $\Delta \equiv 1$.

[11] Shown in courses on Lie groups.

[12] The Lebesgue measure is regular, locally finite and invariant under translations.

[13] For example $(\mathbb{Z}^n, +)$.

$(\det g)^d$, *which shows that* $F(x) = (\det x)^d$. [14]

- *Note that* $GL_d(\mathbb{R})$ *is not abelian, not discrete and not compact, but nevertheless unimodular. The proof of right-invariance is analogous to that of left-invariance.* [15]

Example 6 (Part of Example C.5)**.**

- *Affine translations:*
$$G := \left\{ \begin{pmatrix} a & b \\ & 1 \end{pmatrix} : a \in \mathbb{R}^*, b \in \mathbb{R} \right\}$$
This group is not unimodular, its left-invariant and right-invariant Haar measures are given by:
$$\mu_L := \frac{\mathrm{d}a\,\mathrm{d}b}{|a|^2} \;\; ; \;\; \mu_R := \frac{\mathrm{d}a\,\mathrm{d}b}{|a|} \quad \textit{so} \quad \Delta(a,b) = \frac{1}{|a|}$$

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[14] [Footnote continues at the next page] The Jacobian of the coordinate change is the $d^2 \times d^2$ matrix

$$\begin{pmatrix} g_{11}I & g_{21}I & \cdots & g_{d1}I \\ g_{12}I & g_{22}I & \cdots & g_{d2}I \\ \vdots & \vdots & \ddots & \vdots \\ g_{1d}I & g_{2d}I & \cdots & g_{dd}I \end{pmatrix}$$

where I is the $d \times d$ identity matrix. We have $y_{ij} = \sum_{k=1}^d g_{ik}x_{kj}$ and hence $\frac{\partial y_{ij}}{\partial x_{ab}} = g_{ia}\delta_{jb}$. After identifying y, x with vectors of length d^2 we find that the Jacobian is the matrix above. By an even number of row and column exchanges this matrix can be transformed into

$$\begin{pmatrix} g^t & & & \\ & g^t & & \\ & & \ddots & \\ & & & g^t \end{pmatrix}$$

which has determinant $(\det g)^d$. This implies that m_G as defined above is invariant under coordinate changes of the form $y = g \bullet x$.

[15] This time the Jacobian of the coordinate transformation $z = x \bullet g$ is

$$\begin{pmatrix} g & & & \\ & g & & \\ & & \ddots & \\ & & & g \end{pmatrix}$$

as we have $z_{ij} = \sum_{k=1}^d x_{ik}g_{kj}$ and hence $\frac{\partial y_{ij}}{\partial x_{ab}} = g_{jb}\delta_{ia}$.

The proof uses the Ansatz from the last example. [16]

Example 7 (Part of Example C.5)**.**

- *The following observation shows us what the Haar measures on $G := (\mathbb{R}\backslash\{0\}, \cdot)$ are. We notice that for all $a \in G$:*

$$\int \frac{f(ax)}{|x|}\,\mathrm{d}x = \int \frac{f(x)}{|x|}\,\mathrm{d}x$$

This shows that up to scaling:

$$\mathrm{d}m_G(x) := \frac{\mathrm{d}x}{|x|}$$

- *Similarly, for $G := (\mathbb{C}\backslash\{0\}, \cdot)$ writing $G \ni z = x + iy$:* [17]

$$\mathrm{d}m_G(z) = \frac{\mathrm{d}x\,\mathrm{d}y}{x^2 + y^2}$$

Haar measures can be used to take G-invariant averages. For example, if G has a linear structure and a scalar product $\langle\cdot,\cdot\rangle$ we can use a Haar measure to define a G-invariant scalar product:

$$(x, y) := \int \langle gx, gy\rangle\,\mathrm{d}m_G(g)$$

5 Nice fact

Here we prove that the set $\mathcal{O}$ in Proposition 3 is open (Proposition 9). We use this to derive that $m_{GL_d(\mathbb{R})}(SL_d(\mathbb{R})) = 0$, which follows from Corollary 11.

In the proof of Proposition 9 we use the following lemma. [18]

[16] The Jacobian of the coordinate transformation $y = g \cdot x$ with $g = (a, b), x = (c, d)$ is

$$\begin{pmatrix} a & \\ & a \end{pmatrix}$$

with determinant a^2, which explains the form of the left-invariant Haar measure. For the right-invariant Haar measure we use that the Jacobian of the coordinate transformation $y = x \cdot g$ with $g = (a, b), x = (c, d)$ is

$$\begin{pmatrix} a & \\ b & 1 \end{pmatrix}$$

with determinant a.

[17] It is easy to see invariance under $z \mapsto az$ for $a \in \mathbb{R}_{>0}$ and $a = e^{i\theta}, \theta \in \mathbb{R}$.

[18] The proof of the lemma uses that continuous functions on compact sets are uniformly continuous. In order to simplify the use of uniform compactness metrisable spaces are used. On topological groups it is possible to define uniform continuity without using a metric. Instead of the uniform choice of $\delta > 0$ such that for all $x \in G$ and $y \in B_\delta(x)$ we have the desired inequality, we can fix a neighbourhood B of the identity and translate it using the group structure to get a neighbourhood xB of x for all $x \in G$. That neighbourhood takes the role of $B_\delta(x)$. This is explained in

Lemma 8 (Special case of Lemma 8.7 p. 247)**.** *Let G be a σ-locally compact metrisable group acting continuously on a locally compact, σ-compact, metrisable space X. Let μ be a locally finite measure on X that is invariant under G. Then, for $p \in [1, \infty)$ and $f \in C_c(X) \subset L^p_\mu(X)$ we get for all $g \in G$ a function $U_g f \in L^p_\mu(X)$ defined by*

$$(U_g f)(x) := f(g^{-1} \cdot x)$$

as there holds $\|U_g f\|_p = \|f\|_p$ by G-invariance of μ. Moreover, the map

$$G \to L^p_\mu(X) : g \mapsto U_g f$$

is continuous with respect to the $\| \cdot \|_p$ norm.

The proof can be found in the reference, or in next week's class.

Proposition 9 (Part of Corollary 8.6 pp. 246–247)**.** *Let G be a σ-locally compact metrisable group with left-invariant Haar measure m_G. Then for any two Borel sets $B_1, B_2 \in \mathcal{B}_G$ with $m_G(B_1), m_G(B_2) > 0$ we have that $\mathcal{O} := \{g \in G : m_G(gB_1 \cap B_2) > 0\}$ is open and non-empty.*

Proof. We have shown $m_G(\mathcal{O}) > 0$ in Proposition 3.

It rests to show that $\mathcal{O}$ is open. Let $g \in \mathcal{O}$. By σ-compactness of G we can write:

$$B_1 = \cup_{n \in \mathbb{N}} A_n$$

for A_n all having compact closure and hence finite measure. We get:

$$\exists n \in \mathbb{N} : 0 < m_G(gA_n \cap B_2)$$

Moreover, for any $g_1 \in G$:

$$m_G(g_1 A_n \cap B_2) = \int \chi_{g_1 A_n} \chi_{B_2} \, \mathrm{d}m_G = \int \chi_{A_n}(g_1^{-1} h) \chi_{B_2}(h) \, \mathrm{d}m_G(h)$$

Writing $f := \chi_{A_n}$ we have:

$$\begin{aligned} |m_G(gA_n \cap B_2) - m_G(g_1 A_n \cap B_2)| &\leq \left| \int (f(g^{-1}h) - f(g_1^{-1}h)) \chi_{B_2}(h) \, \mathrm{d}m_G(h) \right| \\ &\leq \|f(g^{-1} \cdot) - f(g_1^{-1} \cdot)\|_1 \end{aligned}$$

It now follows from Lemma 8 that this is arbitrarily small for g_1 close enough to g. In particular, it is smaller than $m_G(gA_n \cap B_2)$, so $0 < m_G(g_1 A_n \cap B_2) \leq m_G(g_1 B_1 \cap B_2) \Rightarrow g_1 \in \mathcal{O}$, for g_1 close enough to g. $\square$[19]

[19] Actually, the statement also holds for $\{g \in G : m_G(B_1 g \cap B_2) > 0\}$. Using that $m_G(B_1^{-1}), m_G(B_2^{-1}) > 0$ by Proposition 3 and that

$$\{g \in G : m_G(B_1 g \cap B_2) > 0\} = \{h \in G : m_G(hB_1^{-1} \cap B_2^{-1}) > 0\}^{-1}$$

we find that $\{g \in G : m_G(B_1 g \cap B_2) > 0\}$ is open and non-empty.

Corollary 10. *If $B \in \mathcal{B}_G$ satisfies $m_G(B) > 0$, then $BB^{-1} := \{b_1 b_2^{-1} : b_1, b_2 \in B\}$ is a neighbourhood of e with non-empty interior.*

Proof.

$$[gB \cap B \neq \emptyset] \quad \Leftrightarrow \quad [\exists b_1, b_2 \in B : gb_1 = b_2] \Leftrightarrow [g = b_2 b_1^{-1} \in BB^{-1}]$$

Hence $BB^{-1} \supset \mathcal{O}$, where $\mathcal{O}$ is the non-empty, open set from in Propositions 3 and 9.

Corollary 11. *Let G be a locally compact topological group with left-invariant Haar measure m_G. Then all subgroups $B \subset G$ with empty interior have zero Haar measure.*

Proof. The fact that B is a subgroup implies that $BB^{-1} = B$, which has empty interior in G. The claim now follows from Corollary 10: if B has positive Haar measure, then the corollary shows it has non-empty interior, which is a contradiction. □

An example of this situation is $SL_d(\mathbb{R}) \subset GL_d(\mathbb{R})$ with the topology inherited from $\mathbb{R}^{d^2}$. With Corollary 11 we find $m_{GL_d(\mathbb{R})}(SL_d(\mathbb{R})) = 0$.

6 Haar measures on $SL_d(\mathbb{R})$

We look at $G := SL_d(\mathbb{R}) \subset \mathbb{R}^{d^2}$ with the relative topology.

Theorem 12. *Up to a constant, the Haar measure of a set $A \subset SL_d(\mathbb{R})$ is given by the volume of the cone in $\mathbb{R}^{d^2}$ between A and the origin. More precisely, up to a constant Haar measures are given by:*

$$\forall A \in \mathcal{B}_G : \lambda_{\mathbb{R}^{d^2}}(\cup_{t\in[0,1]} tA) =: \nu(A)$$

[20]*where $\lambda_{\mathbb{R}^{d^2}}$ is the Lebesgue measure.*

Sketch. For disjoint sets in $SL_d(\mathbb{R})$ the cones they span are disjoint as well, [21] so it follows that ν is a measure. For compact A (closed and bounded) the cone is closed an bounded as well, hence compact and of finite Lebesgue measure, so ν is locally finite.
By definition of the relative topology on $SL_d(\mathbb{R})$ it follows for non-empty open sets $A \in \mathcal{B}_G$ that each point in $\frac{1}{2}A$ has positive distance to the boundary. [22] After reducing the radius we find a ball that is contained in the cone spanned by A, which implies that $\nu(A) > 0$, so ν is positive on non-empty open sets.
For left-invariance we can use the example above. We found that for the transformation $y = g \bullet x$ we have $\det \mathrm{Jac} = (\det g)^d$, which in our case is 1. Hence the Lebesgue measure of $\cup_{t\in[0,1]} tA$ is preserved under $g \in SL_d(\mathbb{R})$.

[20]Note that the union $\cup_{t\in[0,1]} tA$ is disjoint, because the determinant of each element in tA equals t^d.

[21]For $A = \sqcup_{n\in\mathbb{N}} A_n$ we have $\sqcup_{t\in[0,1]} tA = \sqcup_{n\in\mathbb{N}} \sqcup_{t\in[0,1]} tA_n$.

[22]Per definition $\exists B \subset \mathbb{R}^{d^2}$ open such that $A = B \cap SL_d(\mathbb{R})$. Each point in A lies in B, which contains an open ball around that point. Intersecting with A we find that each point in A has positive distance to the boundary.