8.6. Uniform convexity

Definition. Let $(X, \|\cdot\|_X)$ be a Banach space and let $S = \{x \in X \mid \|x\|_X = 1\}$ be the unit sphere in X. The space $(X, \|\cdot\|_X)$ is called *uniformly convex* if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x, y \in S: \quad \|x - y\|_X > \varepsilon \;\Rightarrow\; \left\|\frac{x+y}{2}\right\|_X < 1 - \delta.$$

Remark. Uniform convexity is not to be confused with *strict convexity* defined in problem 8.4.

(a) Prove that Hilbert spaces are uniformly convex.

(b) Provide an example of a Banach space which is not uniformly convex.

. . . .

Solution.

(a) Let $(H, (\cdot,\cdot))$ be a Hilbert space. Let $\varepsilon > 0$. For all $x, y \in H$ with $\|x\| = 1 = \|y\|$ and $\|x - y\| > \varepsilon$, the parallelogram identity (see problem 1.2.) implies

$$\left\|\frac{x+y}{2}\right\|^2 = 2\left\|\frac{x}{2}\right\|^2 + 2\left\|\frac{y}{2}\right\|^2 - \left\|\frac{x-y}{2}\right\|^2 < \frac{1}{2} + \frac{1}{2} - \frac{\varepsilon^2}{4}$$
$$\Rightarrow \left\|\frac{x+y}{2}\right\| \le \left(1 - \frac{\varepsilon^2}{4}\right)^{\frac{1}{2}}.$$

(b) $(L^\infty(\mathbb{R}), \|\cdot\|_{L^\infty})$ is not uniformly convex: Consider the characteristic functions $u = \chi_{[0,1]}$ and $v = \chi_{[t,1+t]}$ and $\varepsilon = \frac{1}{2}$. For any $0 < t < 1$, one has $\|u\|_{L^\infty} = 1 = \|v\|_{L^\infty}$ and $\|u - v\|_{L^\infty} = 1 > \varepsilon$, but $\|\frac{1}{2}(u + v)\|_{L^\infty} = 1$.

In analogy to the first example, the finite-dimensional Banach space $(\mathbb{R}^2, \|\cdot\|_\infty)$, where we define $\|p\|_\infty := \max\{|p_1|, |p_2|\}$ for every $p = (p_1, p_2) \in \mathbb{R}^2$, is not uniformly convex: Let $x = (1, 1)$ and $y = (1, 0)$. Then $\|x\|_\infty = 1 = \|y\|_\infty$ and $\|x - y\|_\infty = 1$ but $\|\frac{1}{2}(x + y)\|_\infty = \|(1, \frac{1}{2})\|_\infty = 1$.