7.1. Derivative operator on different spaces

"If you do not know what to do, integrate by parts." [A. Carlotto after S. T. Yau]

Remark. In part (b), the inclusion $C^1([0,1]) \subset L^2([0,1])$ is to be understood as $C^1([0,1]) = \{f \in L^2([0,1]) \mid f \text{ has a continuously differentiable representative}\}$.

7.2. Complementing subspaces of finite dimension or codimension

For (a), consider a basis $e_1, \ldots, e_n$ of U and corresponding linear functionals $f_i \colon U \to \mathbb{R}$ which can be extended to X by the Hahn-Banach Theorem. Use them to construct a projection onto U and apply the results of problem 5.6.

Part (b) is similar, but instead of extending, precompose functionals $X/U \to \mathbb{R}$ with the quotient map $\pi \colon X \to X/U$.

7.3. Dense kernel

A linear map $f \colon X \to \mathbb{R}$ is not continuous if there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ in X with $\|x_k\|_X = 1$ but $|f(x_k)| \to \infty$ as $k \to \infty$.

7.4. Attaining the distance from the kernel

Show $\dim(X/N) = 1$ and conclude that every $x \in X$ is of the form $x = tx_0 + y$ with uniquely determined $t \in \mathbb{R}$ and $y \in N$.

7.5. Not attaining the distance from the kernel

The solution is very short if you apply the results from problems 4.5 and 7.4.

7.6. Unique extension of functionals on Hilbert spaces

Is f closable? Reduce to the case that Y is a closed subspace. If Y is closed, the Riesz representation theorem applies.

7.7. Distance from convex sets in Hilbert spaces

By a translation you can reduce to the case $x = 0$ which eases notation.

Convexity of Q means in particular that $x_1, x_2 \in Q \Rightarrow \frac{x_1+x_2}{2} \in Q$.

Apply the parallelogram identity from problem 1.2.