8.1. A result by Lions-Stampacchia

Check that the Lax-Milgram Theorem applies to $a(\cdot,\cdot)$ and prove that there exists a unique $x_0 \in H$ with $J(x) = a(x - x_0, x - x_0) - a(x_0, x_0)$.

For the first inequality, show that $(H, a(\cdot,\cdot))$ is a Hilbert space and apply the result of problem 7.7 (b) in $(H, a(\cdot,\cdot))$. Note that convexity of K is used here.

For the second inequality, combine and exploit convexity of K and the minimality property of y_0.

8.2. Duality of sequence spaces

(a) Recall problem 3.3.

(b) Follow the proof that $(L^p(\Omega))^*$ is isometrically isomorphic to $L^q(\Omega)$ for $\frac{1}{p} + \frac{1}{q} = 1$ (Satz 4.4.1) and construct an isometry $\Psi\colon \ell^1 \to c_0^*$ analogously. To prove surjectivity, show that every functional $f \in c_0^*$ is determined by its values on the elements $e^{(k)} = (e_n^{(k)})_{n\in\mathbb{N}} \in c_0$, where $e^{(k)} = (0, \ldots, 0, 1, 0, \ldots)$ has the 1 at k-th position.

(c) Find a bijective linear map $T\colon c \to c_0$ and compose functionals in c_0^* with T to obtain functionals in c^*.

8.3. Projection to convex sets

Prove that if $y \in K$, then $(Px_0 - x_0, y - Px_0)_H \geq 0$ for any $x_0 \in H$. You can use this twice in (a) and once in (b).

(In fact, the inequality above is a special case of the second inequality shown in problem 8.1 for $a(\cdot,\cdot) = (\cdot,\cdot)_H$ with $y_0 = Px_0$.)

8.4. Strict Convexity

(a) Suppose two different elements in X^* have the property in question. Divide by $\|x\|_X^2$ such that everything has norm 1 and apply strict convexity of X^* with $\lambda = \frac{1}{2}$.

(b) You can find a finite-dimensional example.

8.5. Functional on the span of a sequence

Read the inequality in (ii) as $|\tilde{\ell}(y)| \leq \gamma \|y\|_X$ for some functional $\tilde{\ell}$ defined on a suitable subspace of X which can be extended to X using the Hahn-Banach Theorem.