10.1. Project: The weak topology is not metrizable

(a) With a metric, a countable neighbourhood basis can be constructed explicitly.

(b) How does any open set in the weak topology which contains the origin look like?

(c) Prove that if $\varphi\colon X \to \mathbb{R}^n$ and $f\colon X \to \mathbb{R}$ are linear maps such that $\ker \varphi \subset \ker f$, then there exists a linear map $F\colon \mathbb{R}^n \to \mathbb{R}$ such that $f = F \circ \varphi$. Then choose φ wisely.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\varphi"'] & & \mathbb{R} \\ & \mathbb{R}^n \arrow[ur, dashed, "F"'] & \end{tikzcd}$$

(d) If $\{A_\alpha\}_{\alpha\in\mathbb{N}}$ is a countable neighbourhood basis of $0 \in X$ in (X, τ_{w}) then each A_α contains one of the neighbourhoods of the neighbourhood basis constructed in (b). Use this fact to construct a countable set $F \subset X^*$ such that every $f^* \in X^*$ is a linear combination of finitely many elements in F. Apply part (c) for this last step.

(e) Recall that $(X^*, \|\cdot\|_{X^*})$ is always complete.

10.2. Sequential closure

(a) Argue by contradiction.

(b) Aim at finding a set $\Omega \subset \ell^2$ such that $(0) := (0, 0, \dots) \in \overline{\Omega}_{\mathrm{w}}$ but no sequence in Ω converges weakly to zero: $(0) \notin \overline{\Omega}_{\text{w-seq}}$. Notice that such $\Omega \subset \ell^2$ must be unbounded. Recall what $(0) \in \overline{\Omega}_{\mathrm{w}}$ and $(0) \notin \overline{\Omega}_{\text{w-seq}}$ both mean by definition.

10.3. Convex hull

(a) Prove the two inclusions. One inclusion follows by showing that the right hand side is convex.

(b) According to Problem 10.2 (a), arbitrary subsets $\Omega \subset X$ satisfy the inclusions

$$\Omega \subset \overline{\Omega} \subset \overline{\Omega}_{\text{w-seq}} \subset \overline{\Omega}_{\mathrm{w}},$$

where $\overline{\Omega}$ denotes the closure in the norm-topology, $\overline{\Omega}_{\text{w-seq}}$ the weak-sequential closure and $\overline{\Omega}_{\mathrm{w}}$ the closure in the weak topology. Mazur's Lemma is based on the fact that for convex sets, the closure with respect to the norm-topology agrees with the closure in the weak topology (Satz 4.6.2).

(c) Show first that

$$\operatorname{conv}(A \cup B) = \bigcup_{\substack{s,t \geq 0 \\ s+t=1}} (sA + tB)$$

and then argue that the right hand side is compact.

10.4. Non-compactness

(a) Find a sequence of bounded functions $f_n \colon [0,1] \to \mathbb{R}$ such that for *any* pair $n, m \in \mathbb{N}$ with $n \neq m$ the difference $f_n - f_m$ is non-zero on a set of measure $\frac{1}{2}$.

(b) The simplest sequence works.

10.5. Separability

Are subsets of separable sets separable?
How can a countable dense set in S be "scaled" to a countable dense set in X?

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