11.1. Dual Operators

Every statement follows directly from the property which characterises dual operators. You can apply (a) and (b) to show (c).

11.2. Isomorphisms and Isometries

(a) Apply 11.1 (c).

(b) Due to part (a) it suffices to show $\|T^*y^*\|_{X^*} = \|y^*\|_{Y^*}$ for every $y^* \in Y^*$.

(c) Combine part (a) respectively (b) with the result of problem 11.1 (d).

(d) Apply part (a) twice and reuse the result of problem 11.1 (d).

11.3. Minimal Energy

(a) Show that the linear operator $T\colon L^2(\Omega) \to L^2(\Omega)$ mapping $f \mapsto Tf$ given by

$$(Tf)(x) = \int_\Omega g(x-y)f(y)\,\mathrm{d}y$$

is well defined. Analyse the image of a weakly convergent sequence under T. Recall that the L^2-scalar product satisfies the continuity property proven in problem 9.6 (b).

(b) Verify the requirements needed for the direct method in the calculus of variations.

11.4. Compact Operators

(a) A sequence in $\overline{T(B_1(0))}$ can be approximated by a sequence in $T(B_1(0))$ which is the image of a (bounded) sequence in $B_1(0)$.

(b) Use (a) and a diagonal sequence argument.

(c) In finite dimensions, sets which are bounded and closed are compact.

(d) Use (a) and continuity of T respectively S.

(e) Apply the Eberlein–Šmulian theorem.