12.1. Integral operators

(a) Apply Hölder's inequality and Fubini's theorem.

(b) By Fubini's theorem, $k(x,\cdot) \in L^2(\Omega)$ for almost every $x \in \Omega$. Apply the dominated convergence theorem in $L^2(\Omega)$. You may use problem 11.4 (e) to conclude.

12.2. Uniform subconvergence

Apply the Arzelà–Ascoli theorem.

12.3. Multiplication operators on complex-valued sequences

(a) Computing $\|Te_k\|_{\ell^2_{\mathbb{C}}}$ for $e_k = (0,\dots,0,1,0,\dots) \in \ell^2_{\mathbb{C}}$ yields a lower bound on $\|T\|$.

(b) Compute the adjoint operator T^* explicitly.

(c) To prove "$\Leftarrow$", show that the components $x_n^{(k)}$ of any bounded sequence $(x^{(k)})_{k\in\mathbb{N}}$ in $\ell^2_{\mathbb{C}}$ converge in $\mathbb{C}$ as $k \to \infty$.

12.4. A compact operator on continuous functions

(a) Compute the integral

$$\int_a^x \frac{1}{\sqrt{x-t}}\,dx.$$

(b) Apply the Arzelà–Ascoli theorem.

(c) Recall the definition of spectral radius and appeal to part (a).

12.5. A multiplication operator on square-integrable functions

(a) Computing $\|T\chi\|_{L^2([a,b];\mathbb{C})}$ for characteristic functions χ supported near a respectively b yields a lower bound on $\|T\|$.

(b) Recall that $Tf = \lambda f$ in $L^2([a,b])$ means $(Tf)(x) = \lambda f(x)$ for *almost* all $x \in [a,b]$.

(c) Part (b) implies that the operator $(\lambda - T)$ is injective for any $\lambda \in \mathbb{C}$. For which $\lambda \in \mathbb{C}$ is the operator $(\lambda - T)\colon L^2([a,b];\mathbb{C}) \to L^2([a,b];\mathbb{C})$ also surjective? As an example, compute f such that $(\lambda - T)f = 1 \in L^2([a,b])$.