13.1. Definitions of resolvent set

Given $\lambda \in \tilde{\rho}(A)$, prove that $(\lambda - A)\colon D_A \to X$ is surjective.

13.2. Unitary operators

(a) Why do isometries $T \in L(H,H)$ satisfy $\langle Tx, Ty\rangle_H = \langle x, y\rangle_H$ for every $x, y \in H$?

(b) Recall that the spectral radius of any operator T is bounded from above by $\|T\|$ and recall the statements of Satz 6.5.3.i and Satz 2.2.7.

13.3. Integral operators revisited

Recall the properties of K proven in problem 12.1. Check that K and A are both self-adjoint operators. Conclude as in the proof of Beispiel 6.5.2.

13.4. Resolvents and spectral distance

(a) Prove $R_\lambda^* = R_{\overline{\lambda}}$ and use that resolvents to different values commute (Satz 6.5.2).

(b) Argue, that it suffices to show the following implication for any $\alpha \in \mathbb{C}$.

$$\inf_{\beta\in\sigma(B)} |\alpha - \beta| > \|A - B\|_{L(H,H)} \qquad \Rightarrow \ \alpha \in \rho(A).$$

Given $f_\alpha(z) = (\alpha - z)^{-1}$, the spectral mapping theorem implies $f_\alpha(\sigma(B)) = \sigma(f_\alpha(B))$. Show that normal operators R have spectral radius $r_R = \|R\|$. Apply Satz 2.2.7.

13.5. Heisenberg's uncertainty principle

Be pedantic about operator domains.

(a) Apply the Cauchy–Schwarz inequality.

(b) To apply part (a), find symmetric operators $\tilde{A} = A - \lambda$ and $\tilde{B} = B - \mu$ satisfying

$$[A,B] = [\tilde{A},\tilde{B}], \qquad \varsigma(A,x) = \|\tilde{A}x\|_H, \qquad \varsigma(B,x) = \|\tilde{B}x\|_H.$$

(c) Check that $[A, B^n]$ is well-defined and prove $[A, B^n] = niB^{n-1}$ for every $n \in \mathbb{N}$.

(d) The checklist is

$$\begin{aligned}
&\forall f \in D_P := C_0^1([0,1];\mathbb{C}): \quad Pf \in L^2([0,1];\mathbb{C}), && (\Rightarrow P \text{ well-defined})\\
&\forall f \in D_Q := L^2([0,1];\mathbb{C}): \quad Qf \in L^2([0,1];\mathbb{C}), && (\Rightarrow Q \text{ well-defined})\\
&\forall f \in D_{[P,Q]} := D_P \cap D_Q: \quad Qf \in D_P, && (\Rightarrow [P,Q] \text{ well-defined})\\
&\forall f, g \in D_P: \quad \langle Pf, g\rangle_{L^2} = \langle f, Pg\rangle_{L^2}, && (\Rightarrow P \text{ symmetric})\\
&\forall f, g \in D_Q: \quad \langle Qf, g\rangle_{L^2} = \langle f, Qg\rangle_{L^2}, && (\Rightarrow Q \text{ symmetric})\\
&\forall f \in D_{[P,Q]}: \quad ([P,Q]f)(s) = if(s). && (\text{Heisenberg-pair})
\end{aligned}$$