1.1. Equivalent Norms

Definition. Let X be a set. A *metric* on X is a non-negative function $d\colon X \times X \to \mathbb{R}$ which satisfies for all $x, y, z \in X$

$$d(x,y) = 0 \Leftrightarrow x = y, \qquad d(x,y) = d(y,x), \qquad d(x,z) \leq d(x,y) + d(y,z).$$

We say that two metrics d and d' on X are *equivalent* if

$$\exists C > 0 \quad \forall x_1, x_2 \in X: \quad C^{-1} d'(x_1,x_2) \leq d(x_1,x_2) \leq C d'(x_1,x_2).$$

Let X be a vector space over $\mathbb{R}$. A *norm* on X is a non-negative function $\|\cdot\|\colon X \to \mathbb{R}$ which satisfies for all $x, y \in X$ and $\lambda \in \mathbb{R}$

$$\|x\| = 0 \Leftrightarrow x = 0, \qquad \|\lambda x\| = |\lambda| \|x\|, \qquad \|x + y\| \leq \|x\| + \|y\|.$$

We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on X are *equivalent* if

$$\exists C > 0 \quad \forall x \in X: \quad C^{-1} \|x\|' \leq \|x\| \leq C \|x\|'.$$

Recall that a norm $\|\cdot\|$ induces a metric $d_{\|\cdot\|}$ by the formula $d_{\|\cdot\|}(x_1, x_2) = \|x_1 - x_2\|$.

(a) Let X be a finite-dimensional vector space over $\mathbb{R}$. Show that all norms on X are equivalent.

(b) Construct two metrics on $\mathbb{R}^2$ that are *not* equivalent.

(c) Construct a vector space X with two norms $\|\cdot\|$ and $\|\cdot\|'$ that are *not* equivalent.

Hint. Prove that $\|\cdot\|$ and $\|\cdot\|'$ are not equivalent by exhibiting a sequence $(x_n) \subset X$ that converges for $\|\cdot\|$ but not for $\|\cdot\|'$.

1.2. Intrinsic Characterisations

Let V be a vector space over $\mathbb{R}$. Prove the following equivalences.

(a) The norm $\|\cdot\|$ is induced by a scalar product $\langle \cdot, \cdot \rangle$ (in the sense that there exists a scalar product $\langle \cdot, \cdot \rangle$ such that $\forall x \in V: \ \|x\|^2 = \langle x, x \rangle$)

$\Leftrightarrow$ the norm satisfies the *parallelogram identity*, i. e. $\forall x, y \in V:$

$$\|x + y\|^2 + \|x - y\|^2 = 2\big(\|x\|^2 + \|y\|^2\big).$$

Hint. If $\|\cdot\|$ satisfies the parallelogram identity, consider $\langle x, y \rangle := \frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$. Prove $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ first for $\lambda \in \mathbb{N}$, then for $\lambda \in \mathbb{Q}$ and finally for $\lambda \in \mathbb{R}$.

(b) The metric $d(\cdot,\cdot)$ is induced by a norm $\|\cdot\|$ (in the sense that there exists a norm $\|\cdot\|$ such that $\forall x, y \in V: \ d(x,y) = \|x-y\|$)

$\Leftrightarrow$ the metric is *translation invariant* and *homogeneous*, i. e. $\forall v, x, y \in V \ \forall \lambda \in \mathbb{R}$:

$$d(x+v, y+v) = d(x,y),$$
$$d(\lambda x, \lambda y) = |\lambda| d(x,y).$$

1.3. Infinite-dimensional vector spaces and separability

(a) Let $\emptyset \neq \Omega \subset \mathbb{R}^n$ be an open set. Show that $L^p(\Omega)$ is an infinite-dimensional vector space for all $1 \leq p \leq \infty$.

(b) Let $(X, \mathcal{A}, \mu)$ be a measure space. Recall that if X is separable and the measure μ is finite (or, more generally, σ-finite) and if $1 \leq p < \infty$, then the space $L^p(X, \mathcal{A}, \mu)$ is separable. Roughly speaking, in the simple case when $X = (0,1)$, $\mathcal{A} =$ Borel-σ-algebra and $\mu = \mathscr{L}^1$, this relies on the fact that any element in those spaces can be arbitrarily well approximated by a function of the form

$$f = \sum_{i=1}^{k} q_i \, \chi_{B_i} \qquad \text{for } k \in \mathbb{N}, \ B_i := B_{r_i}(x_i), \ q_i \in \mathbb{Q}, \ x_i \in \mathbb{Q} \cap (0,1), \ 0 < r_i \in \mathbb{Q}.$$

Show that instead $\left(L^\infty((0,1)), \|\cdot\|_{L^\infty((0,1))}\right)$ is *not* separable, i. e. it does not contain a countable dense subset.

(Recall that $\|u\|_{L^\infty((0,1))} := \inf\{K > 0 \mid |u(x)| \leq K \text{ for almost every } x \in (0,1)\}$.)