2.1. A metric on sequences

Let $S := \{(s_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{R} : s_n \in \mathbb{R}\}$ be the set of all real-valued sequences.

(a) Prove that

$$d\Big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\Big) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

defines a metric d on S.

(b) Show that (S, d) is complete.

(c) Show that $S_c := \{(s_n)_{n\in\mathbb{N}} \in S \mid \exists N \in \mathbb{N}\, \forall n \geq N : s_n = 0\}$, the space of sequences with compact support, is dense in (S, d).

2.2. A metric on $C^0(\mathbb{R}^m)$

Let $K_1 \subset K_2 \subset \ldots \subset \mathbb{R}^m$ be a family of compact subsets such that $K_n \subset K_{n+1}^\circ$ for every $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} K_n = \mathbb{R}^m$.

(a) Prove that

$$d(f, g) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{\|f - g\|_{C^0(K_n)}}{1 + \|f - g\|_{C^0(K_n)}}$$

defines a metric d on $C^0(\mathbb{R}^m)$.

(b) Show that $(C^0(\mathbb{R}^m), d)$ is complete.

(c) Show that $C^0_c(\mathbb{R}^m)$, the space of continuous functions with compact support, is dense in $(C^0(\mathbb{R}^m), d)$.

Remark. The same conclusions with the same proofs also hold for *any* open set $\Omega \subseteq \mathbb{R}^m$ in place of $\mathbb{R}^m$. The metric d deals with the fact that $C^0(\mathbb{R}^m)$ contains unbounded functions like $f(x) = |x|^2$ for which $\sup_{x\in\mathbb{R}^m} |f(x)| = \infty$.

2.3. Statements of Baire

Definition. Let (M, d) be a metric space and $A \subset M$ a subset. Then, $\overline{A}$ denotes the closure, A° the interior and $A^\complement = M \setminus A$ the complement of A. We call A

- *dense*, if for every ball $B \subset M$, there is an element $x \in B \cap A$.
- *nowhere dense*, if $(\overline{A})^\circ = \emptyset$.
- *meagre*, if $A = \bigcup_{n\in\mathbb{N}} A_n$ is a countable union of nowhere dense sets A_n.
- *residual*, if $A^\complement$ is meagre.

Show that the following statements are equivalent.

(i) Every residual set $\Omega \subset M$ is dense in M.

(ii) The interior of every meagre set $A \subset M$ is empty.

(iii) The empty set is the only subset of M which is open and meagre.

(iv) Countable intersections of dense open sets are dense.

Hint. Show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Use that subsets of meagre sets are meagre and recall that $A \subset M$ is dense $\Leftrightarrow$ $\overline{A} = M$ $\Leftrightarrow$ $(M \setminus A)^\circ = \emptyset$.

Remark. Baire's theorem states that (i), (ii), (iii), (iv) are true if (M, d) is complete.

2.4. Discrete L^p-spaces and inclusions

Let $(x_n)_{n\in\mathbb{N}}$ be a real-valued sequence. Given $1 \le p \le \infty$, we define the norm

$$\left\|(x_n)_{n\in\mathbb{N}}\right\|_{\ell^p} := \begin{cases} \left(\sum_{n\in\mathbb{N}} |x_n|^p\right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \sup\limits_{n\in\mathbb{N}} |x_n|, & \text{if } p = \infty \end{cases}$$

and the space

$$\ell^p := \{(x_n)_{n\in\mathbb{N}} \mid \|(x_n)_{n\in\mathbb{N}}\|_{\ell^p} < \infty\}.$$

Remark. Notice that $\ell^p = L^p(\mathbb{N}, \mathcal{A}, \mu)$, where $\mathcal{A}$ is the σ-algebra of all subsets of $\mathbb{N}$ and μ is the counting measure, i. e. $\mu(M) =$ (number of elements in M). In particular, ℓ^p is a complete space.

Let $1 \le p < q \le \infty$ be given.

(a) Prove the strict inclusion $\ell^p \subsetneq \ell^q$ and the inequality $\|x\|_{\ell^q} \le \|x\|_{\ell^p}$ for all $x \in \ell^p$.

(b) Show that ℓ^p is meagre in $(\ell^q, \|\cdot\|_{\ell^q})$.

Hint. Prove that $A_n = \{x \in \ell^q \mid \|x\|_{\ell^p} \le n\}$ is a closed subset of $(\ell^q, \|\cdot\|_{\ell^q})$ with empty interior.

(c) Use (b) to prove

$$\bigcup_{p\in[1,q[} \ell^p \subsetneq \ell^q.$$

Remark. You can solve (c) without knowing the proof of (b).